Implicit Differentiation

Course Survey closes on Monday
on Canvas under Quizzes

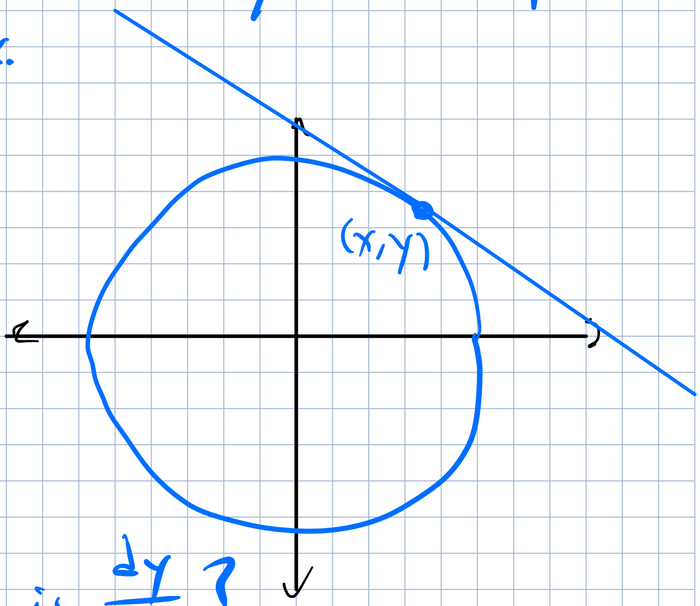
Usually we define a function explicitly

$y = f(x)$ ex: $y = 3x^2 + 5x$

$$\frac{dy}{dx} = 3 \cdot 2x + 5 \cdot 1$$
$$= 6x + 5$$

Sometimes an equation defines y implicitly as a function of x.

Ex: $x^2 + y^2 = 1$.



What is the slope of the tangent line to the unit circle at the point (x,y)?

Same as asking: what is $\frac{dy}{dx}$?

$$x^2 + y^2 = 1$$

Take the derivative of both sides with respect to x.

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[1]$$

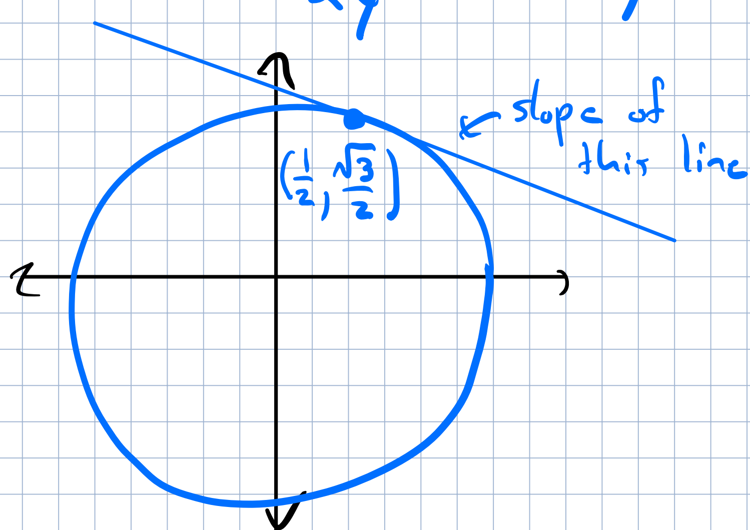
Remember that y is a function of x, so when taking the derivative of an expression involving y, you have to use the chain rule.

$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{\text{CHAIN RULE}} = 0$$

Solve for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}.$$



$$\frac{dy}{dx} = \frac{-x}{y}$$

$$= \frac{-\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}}$$

$$= \frac{-1}{\sqrt{3}}$$

Ex: The function y is given implicitly by the equation $x^2 + 3xy + y^2 = 5$.

Find $\frac{dy}{dx}$.

Take $\frac{d}{dx}[\,-\,]$ of both sides

$$\frac{d}{dx}\left[x^2 + 3xy + y^2\right] = \frac{d}{dx}[5]$$

$$2x + \underbrace{3x \cdot \frac{dy}{dx} + 3 \cdot y}_{\text{PRODUCT RULE applied to } (3x)(y)} + \underbrace{2 \cdot y \cdot \frac{dy}{dx}}_{\text{CHAIN RULE}} = 0$$

Solve for $\frac{dy}{dx}$.

$$3x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 3y$$

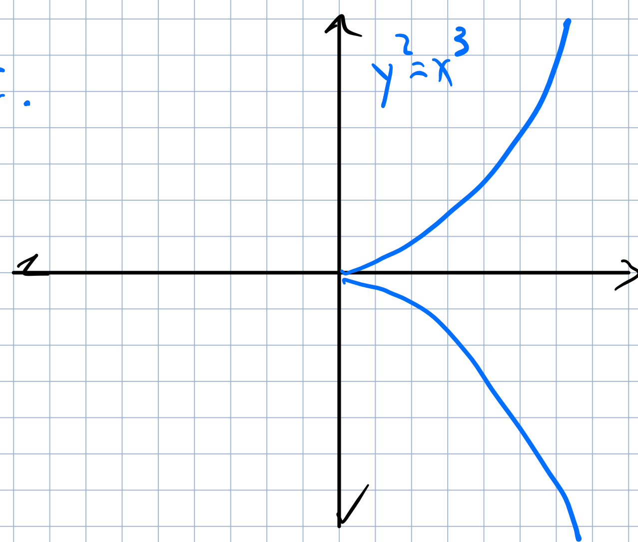
$$(3x+2y)\frac{dy}{dx} = -2x - 3y$$

$$\frac{dy}{dx} = \frac{-2x-3y}{3x+2y}$$

If we wanted to find $\frac{dy}{dx}$ when $(x,y)=(1,1)$,

$$\frac{dy}{dx} = \frac{-2\cdot 1 - 3\cdot 1}{3\cdot 1 + 2\cdot 1} = \frac{-5}{5} = -1.$$

<u>Ex</u>: $y^2 = x^3$. Find $\frac{dy}{dx}$.



$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[x^3\right]$$

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}.$$

This is undefined when $(x,y)=(0,0)$

But $\lim_{x\to 0} \frac{3x^2}{2y}$ is defined. (It's equal to 0.)

<u>Power Rule for Rational Powers</u>

$$y = x^{p/q}$$

p, q whole numbers
$q \neq 0$.

Find $\frac{dy}{dx}$.

Raise both sides to the q^{th} power:

$$y^q = x^p.$$

Take the derivative of both sides.

$$\frac{d}{dx}\left[y^q\right] = \frac{d}{dx}\left[x^p\right]$$

$$q y^{q-1} \frac{dy}{dx} = \underbrace{p x^{p-1}}_{\text{POWER RULE}}$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} \cdot x^{p-1} \cdot x^{p/q - p}$$

$$= \frac{p}{q} \cdot x^{p-1+p/q-p} = \frac{p}{q} x^{p/q - 1}$$

Power Rule for Rational Powers:

$$\boxed{\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}}$$ for any rational number n.