Exam 2 IN CLASS Wednesday, March 9^{th}

6 multiple choice questions

2 short answers ⟵ show your work

What's on it?

Limits

→ Definition

→ Limit Laws

→ Limits at Infinity

→ Continuous Functions & Intermediate Value Thm

→ Sandwich Theorem

Derivatives

→ Definition

→ Power Rule

→ Sum Rule

→ Product Rule

→ Quotient Rule

→ Chain Rule

→ Implicit Differentiation

→ Related Rates

Derivatives of Exponentials

Compute $\frac{d}{dx}\left[a^x\right]$ where a is some number.

$$\frac{d}{dx}\left[a^x\right] = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^h \cdot a^x - a^x}{h} = \underbrace{\left[\lim_{h\to 0} \frac{a^h - 1}{h}\right]}_{\text{some number}} \cdot a^x$$

$$\frac{d}{dx}\left[a^x\right] = (\text{some number}) \cdot a^x$$

The derivative of a^x is proportional to a^x.

What is this number $\lim_{h\to 0} \frac{a^h - 1}{h}$?

Let's think about the special case where

$$a = e \approx 2.71828 \ldots$$

Consider $\lim_{h\to 0} \frac{e^h - 1}{h}$.

h	0.1	0.01	0.001
$\frac{e^h-1}{h}$	1.052...	1.0050...	1.000500...

The number e has the property that

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1.}$$

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$

Now, let $a > 0$ be a positive number.

$$\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[\left(e^{\ln(a)}\right)^x\right]$$

$$= \frac{d}{dx}\left[e^{\ln(a)\cdot x}\right] \overset{\text{CHAIN RULE}}{=} e^{\ln(a)\cdot x} \cdot \frac{d}{dx}\left[\ln(a)\cdot x\right]$$

$$= e^{\ln(a)\cdot x} \cdot \ln(a) = \ln(a) \cdot e^{\ln(a)\cdot x}$$

$$= \ln(a) \cdot a^x$$

$$\boxed{\frac{d}{dx}\left[a^x\right] = \ln(a) \cdot a^x}$$

Ex: $\frac{d}{dx}\left[2^x\right] = \ln(2) \cdot 2^x$

Ex: $\frac{d}{dx}[10^x] = \ln(10) \cdot 10^x$

Ex: $\frac{d}{dx}[e^x] = \cancel{\ln(e)}^{1} \cdot e^x$

Ex: Compute $\frac{d}{dx}\left[\sin(e^x)\right]$.

$f(x) = \sin(x)$ $\quad$ $g(x) = e^x$

$f'(x) = \cos(x)$ $\quad$ $g'(x) = e^x$

$$\frac{d}{dx}[f(g(x))] \overset{\text{CHAIN RULE}}{=} f'(g(x)) \cdot g'(x)$$

$$= \cos(e^x) \cdot e^x.$$

Ex: Compute $\frac{d}{dx}\left[\sqrt{x} \cdot e^x\right]$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad$ $g(x) = e^x$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad$ $g'(x) = e^x$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$= \sqrt{x} \cdot e^x + \frac{e^x}{2\sqrt{x}}.$$

Ex: Compute $\frac{d}{dx}\left[\frac{e^{3x^2+5}}{\tan(x)}\right]$

$f(x) = e^{3x^2+5}$

$g(x) = \tan(x)$

$g'(x) = \sec^2(x)$

To compute $f'(x)$, we need to use the chain rule.

$f_1(x) = e^x$

$g_1(x) = 3x^2+5$

$f_1'(x) = e^x$

$g_1'(x) = 6x$

$$f'(x) = f_1'(g_1(x)) \cdot g_1'(x)$$
$$= e^{3x^2+5} \cdot 6x$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

$$= \frac{\tan(x)\cdot e^{3x^2+5}\cdot 6x - e^{3x^2+5}\cdot \sec^2(x)}{\tan^2(x)}$$

Ex: Compute $\frac{d}{dx}[\ln(x)]$.

$y = \ln(x)$. Want to compute $\frac{dy}{dx}$.

$e^y = x$ implicit differentiation

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

$$xy + 2x + 3x^2 = -8$$

$$\frac{d}{dx}[xy + 2x + 3x^2] = \frac{d}{dx}[-8]$$

$$x\frac{dy}{dx} + y \cdot 1 + 2 + 6x = 0$$

$$x\frac{dy}{dx} = -y - 2 - 6x$$

$$\frac{dy}{dx} = \frac{-y-2-6x}{x}$$