Logarithmic Differentiation

Find the derivative of $y = \ln(x)$.

$$e^y = x.$$

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

Ex: Compute the derivative of $\log_a(x)$.

CHANGE OF BASE FORMULA

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$\frac{d}{dx}[\log_a(x)] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(a)}\right] = \frac{1}{\ln(a)} \cdot \frac{d}{dx}[\ln(x)]$$

$$= \frac{1}{\ln(a)} \cdot \frac{1}{x}.$$

Logarithmic Differentiation

Compute the derivative of a function by taking the ln of both sides and using properties of logarithms.

Ex: Compute the derivative of $y = x^x$

① take ln of both sides ② use properties of logs to simplify

$$\ln(y) = \ln(x^x) = x \cdot \ln(x)$$

③ take derivative of both sides

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[x \cdot \ln(x)]$$

PRODUCT RULE

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}[\ln(x)] + \ln(x) \cdot \frac{d}{dx}[x]$$

$$= x \cdot \frac{1}{x} + \ln(x) \cdot 1$$

$$= 1 + \ln(x)$$

④ plug back in for y

$$\frac{dy}{dx} = (1 + \ln(x)) \cdot y = (1 + \ln(x)) \cdot x^x$$

Ex: Compute the derivative of $y = \sin(x)^x$.

① Take ln of both sides ② use properties of logs to simplify

$$\ln(y) = \ln(\sin(x)^x) = x \cdot \ln(\sin(x))$$

③ Take the derivative of both sides

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[x \cdot \ln(\sin(x))\right]$$

PRODUCT RULE

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}\left[\ln(\sin(x))\right] + \ln(\sin(x)) \cdot \frac{d}{dx}[x]$$

CHAIN RULE

$$= x \cdot \left[\frac{1}{\sin(x)} \cdot \frac{d}{dx}\left[\sin(x)\right]\right] + \ln(\sin(x)) \cdot \frac{d}{dx}[x]$$

$$= x \cdot \frac{1}{\sin(x)} \cdot \cos(x) + \ln(\sin(x)) \cdot 1$$

$$= x\cot(x) + \ln(\sin(x))$$

$$\frac{dy}{dx} = \left[x\cot(x) + \ln(\sin(x))\right] \cdot y$$

④ Plug back in for y

$$\frac{dy}{dx} = \left[x\cot(x) + \ln(\sin(x))\right] \cdot \sin(x)^{x}.$$

Ex: Compute the derivative of $y = \dfrac{\sqrt{x+3} \cdot (x-1)^{3/5}}{(x+7)^{2/3}}$

① Take ln of both sides

$$\ln(y) = \ln\left(\frac{(x+3)^{1/2} \cdot (x-1)^{3/5}}{(x+7)^{2/3}}\right)$$

② Use properties of logs to simplify

$$= \ln\left((x+3)^{1/2}\cdot(x-1)^{3/5}\right) - \ln\left((x+7)^{2/3}\right)$$

$$= \ln\left((x+3)^{1/2}\right) + \ln\left((x-1)^{3/5}\right) - \ln\left((x+7)^{2/3}\right)$$

$$= \frac{1}{2}\ln(x+3) + \frac{3}{5}\ln(x-1) - \frac{2}{3}\ln(x+7)$$

(3) Take the derivative of both sides

(Notice: this is way easier than it would have been at the start b/c Sum Rule is "easier" than the Product & Quotient Rules)

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln(x+3) + \frac{3}{5}\ln(x-1) - \frac{2}{3}\ln(x+7)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x+3}\cdot\frac{d}{dx}[x+3] + \frac{3}{5}\cdot\frac{1}{x-1}\cdot\frac{d}{dx}[x-1] - \frac{2}{3}\cdot\frac{1}{x+7}\cdot\frac{d}{dx}(x+7)$$

$$= \frac{1}{2}\cdot\frac{1}{x+3} + \frac{3}{5}\cdot\frac{1}{x-1} - \frac{2}{3}\cdot\frac{1}{x+7}$$

$$\frac{dy}{dx} = \left[\frac{1}{2(x+3)} + \frac{3}{5(x-1)} - \frac{2}{3(x+7)}\right]\cdot y$$

(4) Plug back in for y

$$\frac{dy}{dx} = \left[\frac{1}{2(x+3)} + \frac{3}{5(x-1)} - \frac{2}{3(x+7)}\right]\cdot\frac{\sqrt{x+3}\cdot(x-1)^{3/5}}{(x+7)^{2/3}}$$

Power Rule, revisited

We've computed $\frac{d}{dx}[x^n]$ when n is any rational number. Using logarithmic differentiation, we can compute it for any real number n.

$$y = x^n$$

(1) Take ln of both sides

(2) Use properties of logs to simplify

$$\ln(y) = \ln(x^n) = n \cdot \ln(x)$$

(3) Take the derivative of both sides

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[n \cdot \ln(x)] = n \cdot \frac{d}{dx}[\ln(x)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x}$$

$$\frac{dy}{dx} = \frac{n}{x} \cdot y$$

(4) Plug back in for y

$$\frac{dy}{dx} = \frac{n}{x} \cdot x^n = n x^{n-1}$$

Most general form of the power rule:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$ for <u>any</u> real number n.