Exam 3 moved to Monday, April 11

Final Project due Monday, April 18 on Canvas

Last time:

Any continuous function $f(x)$ on a closed interval has a maximum and a minimum.

Where could this max or min be?

1. $f'(x) = 0$
2. $f'(x)$ is not defined
3. endpoints of the interval

Ex: Find local maxima and minima of $f(x) = x^3 - x$ on $[-2, 2]$.

1. Where is $f'(x) = 0$?

$$f'(x) = 3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}}.$$

2. Where is $f'(x)$ undefined?

$f'(x)$ is defined everywhere.

(3) Where are the endpoints? $-2, 2$.

$$f(-2) = (-2)^3 - (-2) = -8 + 2 = -6$$

$$f\left(-\sqrt{\tfrac{1}{3}}\right) = \left(-\sqrt{\tfrac{1}{3}}\right)^3 - \left(-\sqrt{\tfrac{1}{3}}\right) = -\tfrac{1}{3}\cdot\sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{1}{3}}$$

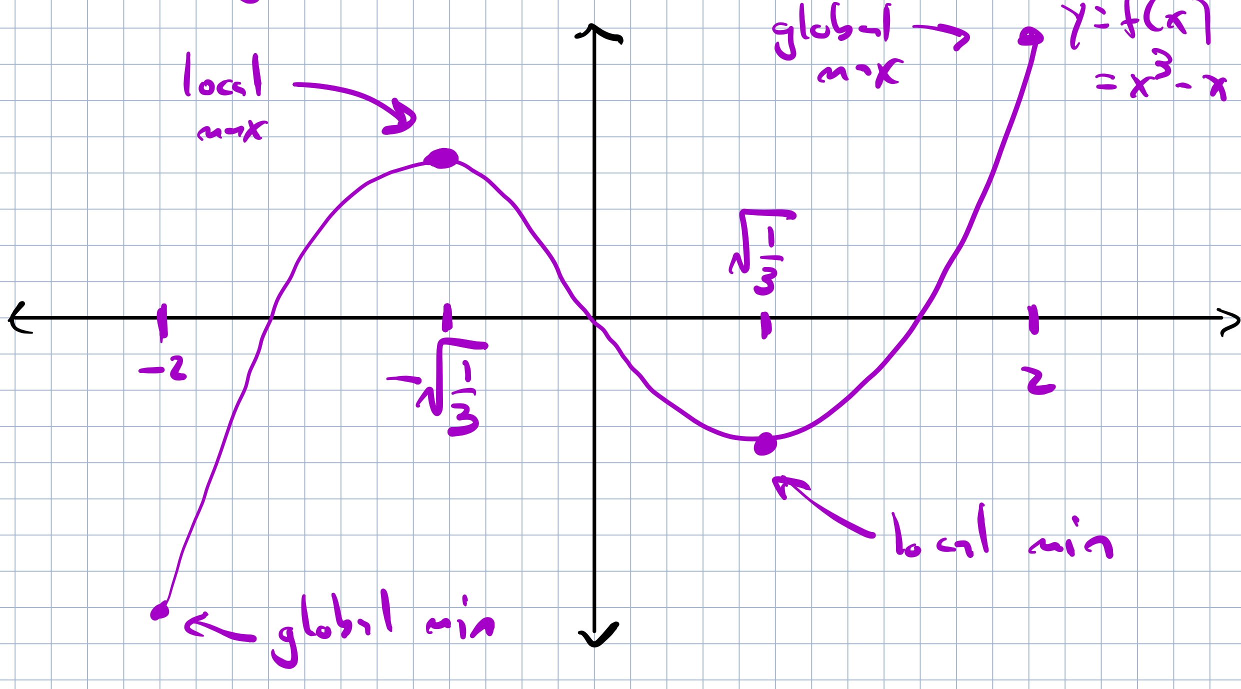
$$= \tfrac{2}{3}\sqrt{\tfrac{1}{3}} \approx 0.38$$

$$f\left(\sqrt{\tfrac{1}{3}}\right) = -\tfrac{2}{3}\sqrt{\tfrac{1}{3}} \approx -0.38$$

$$f(2) = 6$$

Since -6 is the smallest of these 4 values, the global min is at $x=-2$.

Since 6 is the largest of these 4 values, the global max is at $x=2$.

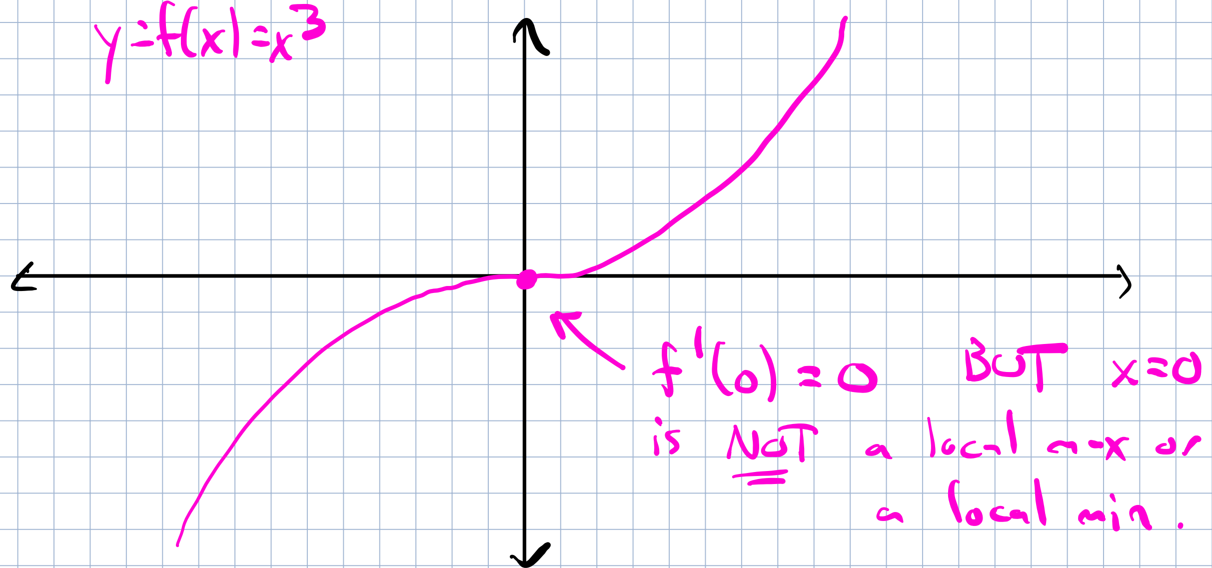


WARNING! You can have $f'(c) = 0$ without having a local max or min at c.

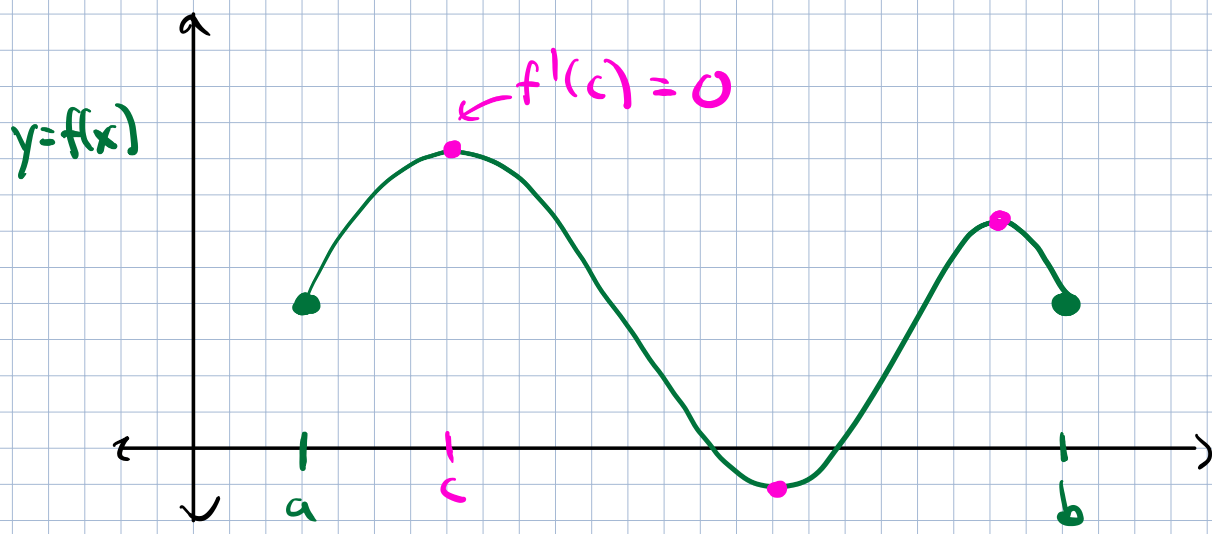
Ex: $f(x) = x^3$.

$$f'(x) = 3x^2 = 0 \text{ when } x^2 = 0$$
$$x = 0.$$

But if $x > 0$, then $f(x) = x^3 > 0$.
if $x < 0$, then $f(x) = x^3 < 0$.



Rolle's Theorem: Let $f(x)$ be a continuous, differentiable on the closed interval $[a, b]$. Suppose that $f(a) = f(b)$. Then there is a number c between a and b such that $f'(c) = 0$.



Why is Rolle's Theorem true?

Since $f(x)$ is continuous on $[a,b]$, we know that $f(x)$ has both a global max and a global min on $[a,b]$.

If the global max or min occurs at a number $a<c<b$, then $f'(c)=0$.

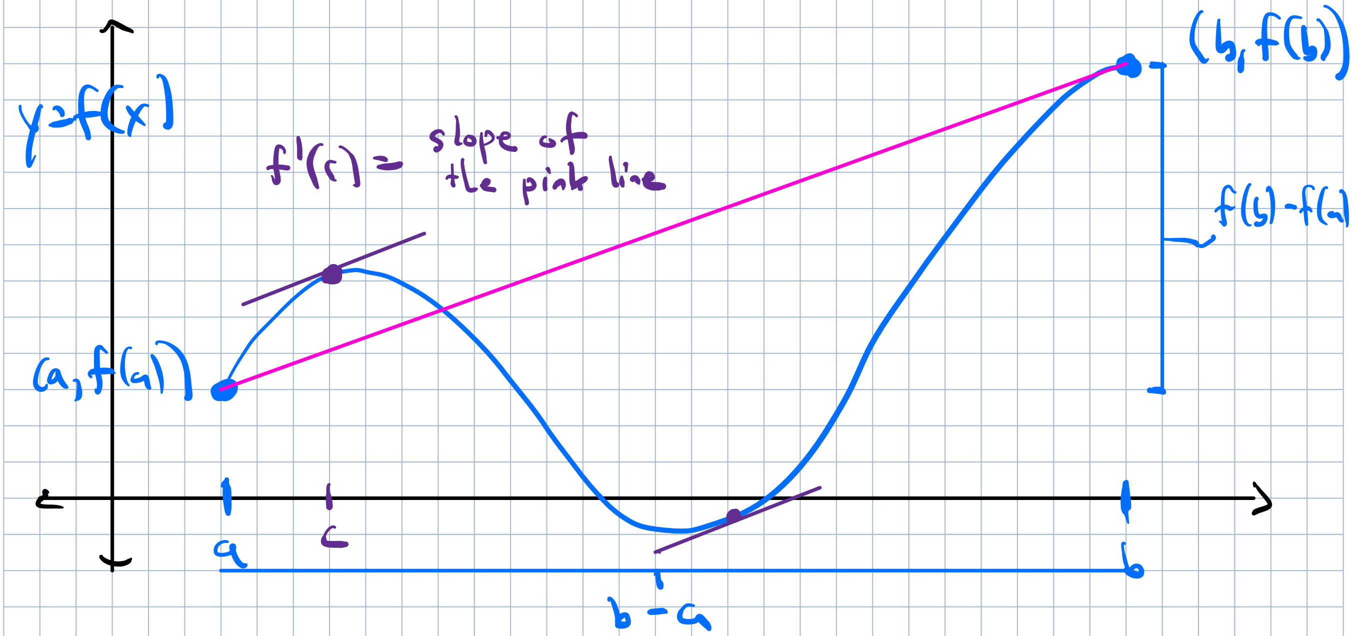
(because c is not an endpoint and $f'(x)$ is defined on the whole interval.)

We're done unless the global max and global min occur at the endpoints.

But, $f(a)=f(b)$. So If the global max occurs at one endpoint and the global min occurs at the other, then the global max is <u>equal</u> to the global min. (Both are equal to $f(a)=f(b)$)

In this case, $f(x)$ is the constant function $f(a)$. Then $f'(c) = 0$ for <u>every</u> c in the interval $[a,b]$.

<u>Mean Value Theorem</u>: Let $f(x)$ be a continuous, differentiable function on the closed interval $[a,b]$. Then there is a number c between a and b such that $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



<u>Ex</u>: If I drive from Lexington to Louisville and my average speed is 70 mph, then at <u>some</u> point along the trip, my instantaneous speed will be 70 mph.

Why is the MVT true?

Consider a new function

$$g(x) = f(x) - \underbrace{\left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)}_{\text{equation for the pink line in the picture above.}}$$

$g(x)$ is continuous and differentiable on $[a,b]$.

$$g(a) = f(a) - \left(\frac{f(b)-f(a)}{b-a}\underbrace{(a-a)}_{0} + f(a)\right)$$

$$= 0.$$

$$g(b) = f(b) - \left(\frac{f(b)-f(a)}{\cancel{b-a}}\cancel{(b-a)} + f(a)\right)$$

$$= f(b) - f(b) = 0.$$

$$g(a) = g(b).$$

So Rolle's Theorem says there is a c where $g'(c) = 0$.

$$g'(x) = \frac{d}{dx}\left[f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)\right]$$

$$= f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

$\square$