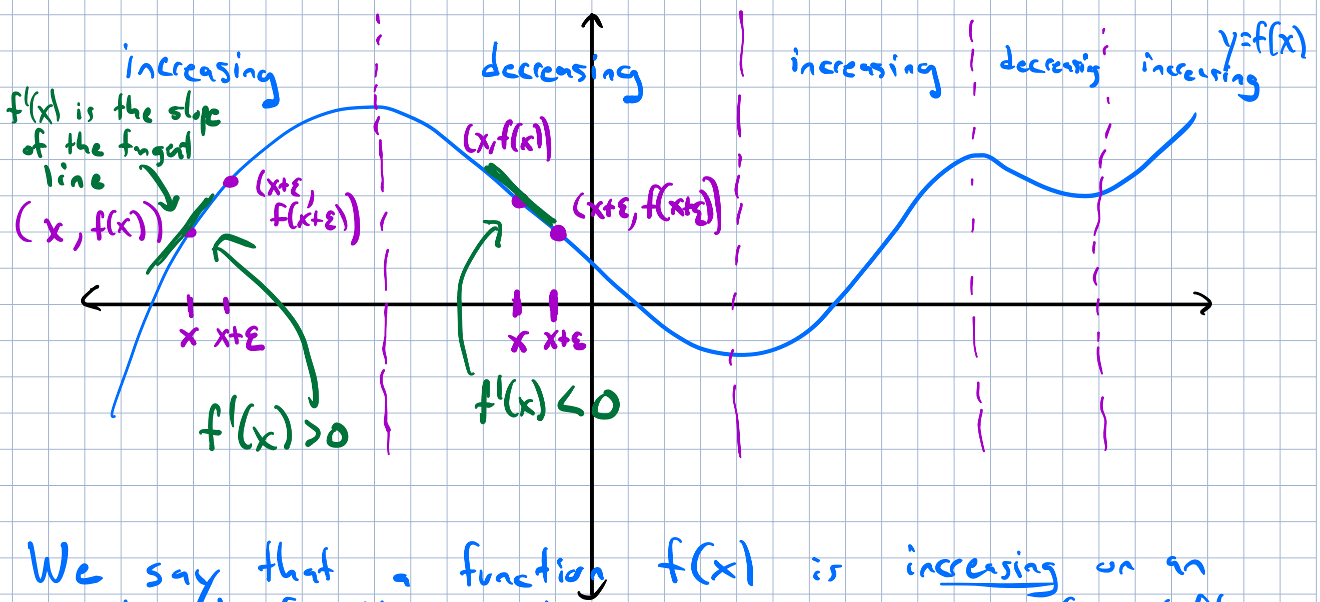
Monotonicity and Concavity

Reminder: Exam 3 in class Monday, April 11

Final project due on Canvas Monday, April 18



We say that a function f(x) is increasing on an interval if the graph goes up as you go from left to right.

More formally, f(x) is increasing at x if, for small positive numbers ε, we have $f(x) < f(x+\varepsilon)$.

Similarly, we say that a function f(x) is decreasing on an interval if the graph goes down as you go from left to right.

More formally, f(x) is decreasing at x if, for small positive numbers ε, we have $f(x) > f(x+\varepsilon)$

A differentiable function f(x) is increasing at x if $f'(x) > 0$.

It is decreasing at x if $f'(x) < 0$.

Ex: Find the intervals on which $f(x)=x^3-x$ is increasing and the intervals on which it is decreasing.

$$f'(x)=3x^2-1$$

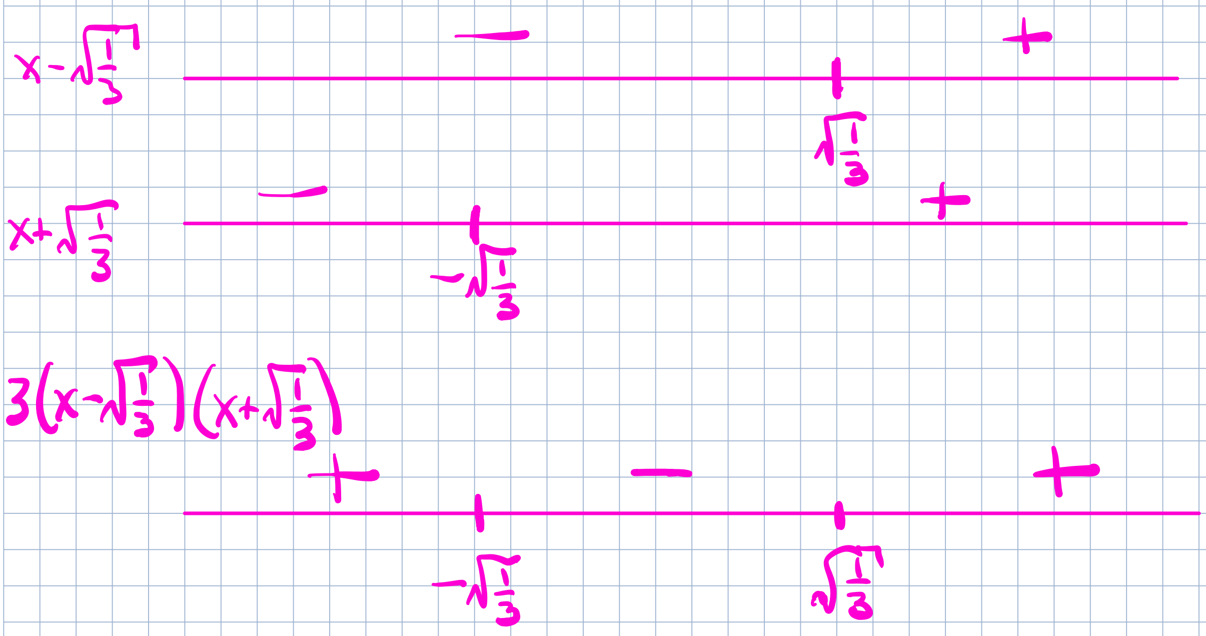
$3x^2-1=0$ when $3x^2=1$

$$x^2=\frac{1}{3}$$

$$x=\pm\sqrt{\frac{1}{3}}.$$

$$f'(x)=3x^2-1=3\left(x-\sqrt{\tfrac{1}{3}}\right)\left(x+\sqrt{\tfrac{1}{3}}\right)$$

Where is $f'(x)>0$?

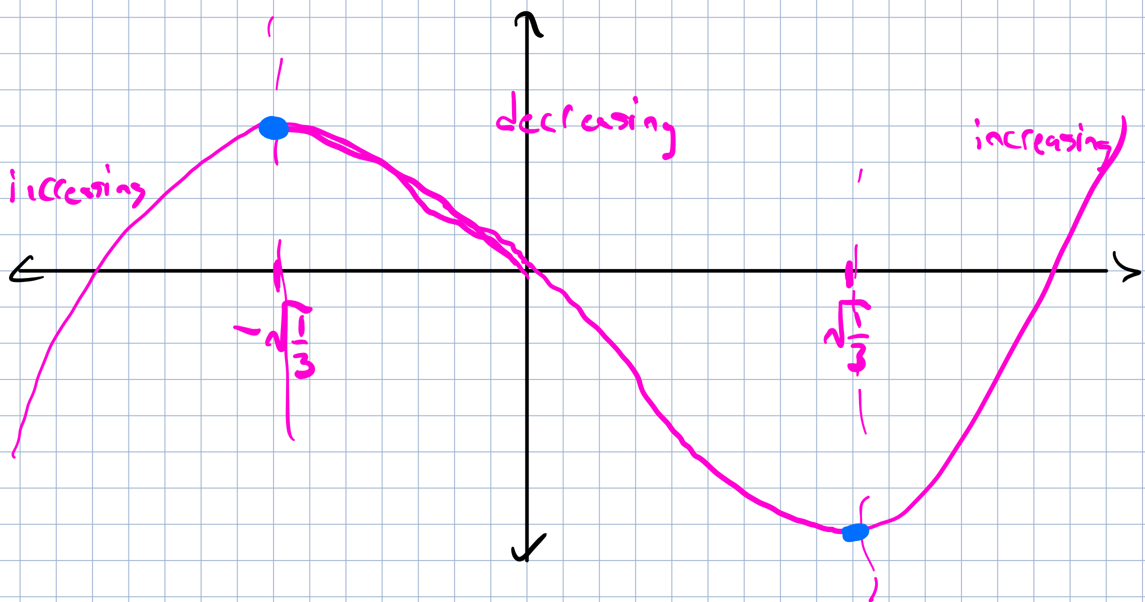


$f'(x)=3\left(x-\sqrt{\tfrac{1}{3}}\right)\left(x+\sqrt{\tfrac{1}{3}}\right)$ is positive $\left(-\infty,\sqrt{\tfrac{1}{3}}\right)$ $\cup\left(\sqrt{\tfrac{1}{3}},\infty\right)$

is negative on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$

$f(x) = x^3 - x$ is increasing on $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$

and is decreasing on $\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{1}{3}}\right)$



A function $f(x)$ has a local maximum at x if it is increasing to the left of x and decreasing to the right of x.

Similarly, it has a local minimum at x if it is decreasing to the left and increasing to the right.

<u>First Derivative Test:</u>

If $f'(x)$ goes from positive to negative at x, then x is a local max of $f(x)$.

If $f'(x)$ goes from negative to positive at x, then x is a local min of $f(x)$.

Ex: Let $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$.

Where is $f(x)$ increasing?

Where is $f(x)$ decreasing?

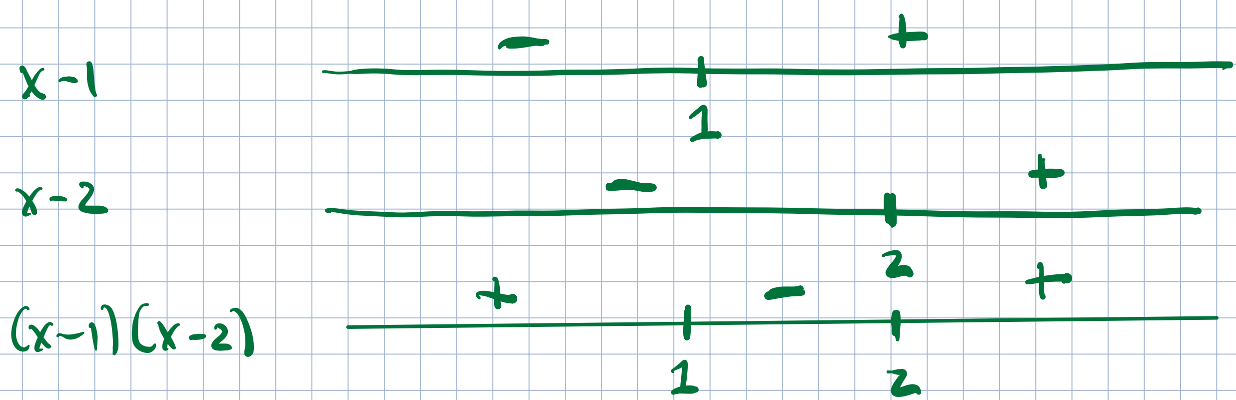
Where are the local maxima and minima?

$$f'(x) = x^2 - 3x + 2$$

$x^2 - 3x + 2 = 0$ when $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$

$$= \frac{3 \pm \sqrt{1}}{2} = \frac{3 \pm 1}{2} = 1, 2$$

$$f'(x) = x^2 - 3x + 2 = (x-1)(x-2).$$



$f'(x) = (x-1)(x-2)$ is positive on $(-\infty, 1) \cup (2, \infty)$ and negative on $(1, 2)$.

$f(x)$ is increasing on $(-\infty, 1) \cup (2, \infty)$ and decreasing on $(1, 2)$

At $x=1$, $f(x)$ goes from increasing to decreasing,
so $x=1$ is a local max.

At $x=2$, $f(x)$ goes from decreasing to increasing,
so $x=2$ is a local min.

Ex: $f(x)=x^3$. Where is $f(x)$ increasing?
Where is $f(x)$ decreasing?

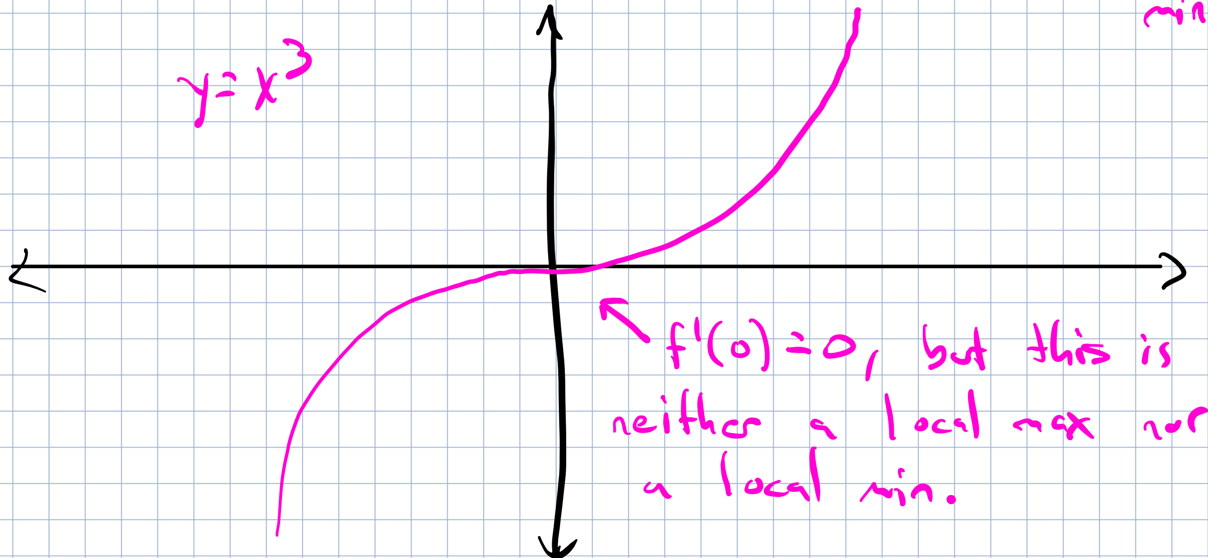
$f'(x)=3x^2$.

$f'(x)=0$ when $x=0$.

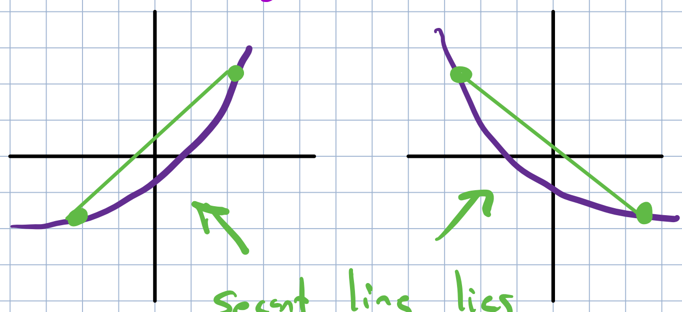
if $x\neq 0$, then $f'(x)=3x^2>0$.

So $f(x)$ is increasing on $(-\infty,0)\cup(0,\infty)$.

So $x=0$ is not a local max or a local min.



Concavity

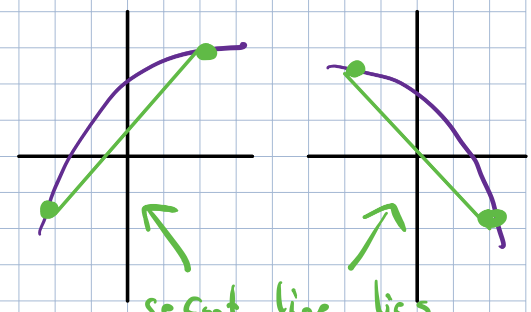


secant line lies <u>above</u> the graph

$f(x)$ is <u>concave up</u>

$= f'(x)$ is increasing

$= f''(x) > 0$



secant line lies <u>below</u> the graph

$f(x)$ is <u>concave down</u>

$= f'(x)$ is decreasing

$= f''(x) < 0$

A twice-differentiable function is <u>concave up</u> at x if $f''(x) > 0$ and it is <u>concave down</u> at x if $f''(x) < 0$.