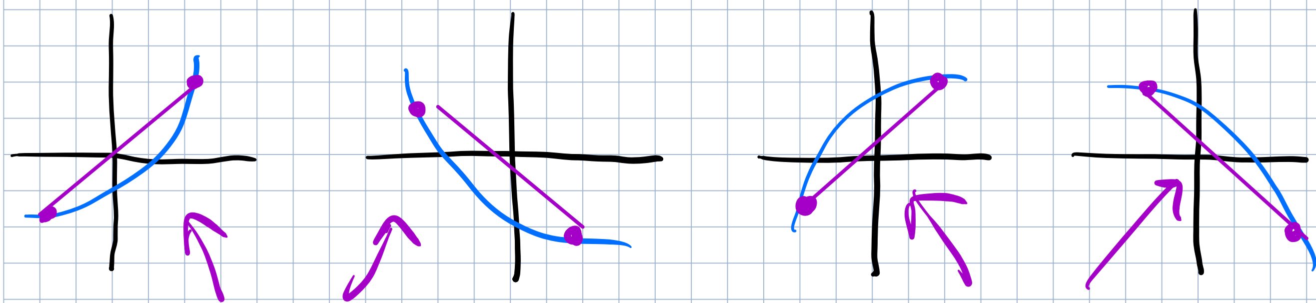
Exam 3 IN CLASS Monday, April 11

Final Project due on Canvas Monday, April 18

Inflection Points



concave up concave down

A function $f(x)$ is <u>concave up</u> at x if

$$f''(x) > 0$$

It is <u>concave down</u> at x if $f''(x) < 0$.

<u>Ex</u>: $f(x) = x^3$. When is it concave up and when is it concave down?

$$f'(x) = 3x^2.$$

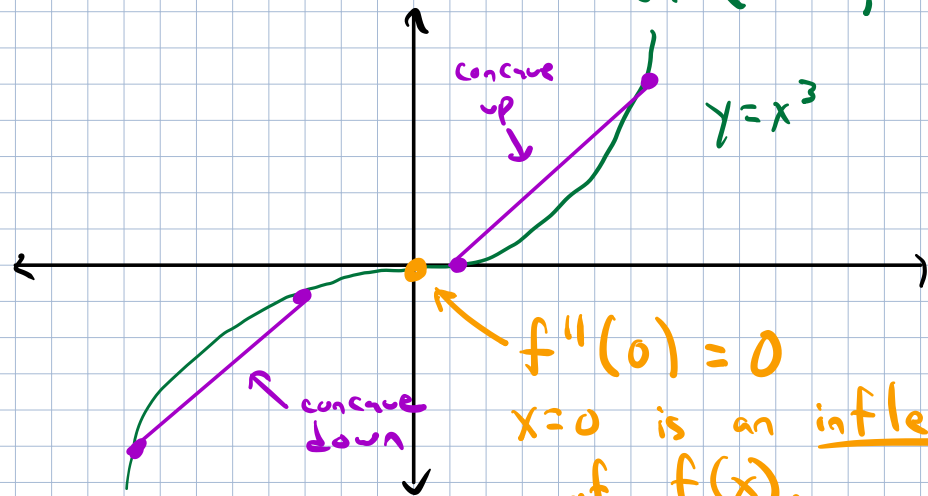
$$f''(x) = 3 \cdot 2x = 6x$$

$$f''(x) = 6x > 0 \quad \text{when} \quad x > 0$$

$$6x < 0 \quad \text{when} \quad x < 0.$$

$f(x) = x^3$ is concave up when $x > 0$ and concave down when $x < 0$.

$f(x) = x^3$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.



$f''(0) = 0$

$x = 0$ is an <u>inflection point</u> of $f(x)$.

A function $f(x)$ has an <u>inflection point</u> at x if $f''(x) = 0$.

<u>Ex</u>: Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 4$.

Find the inflection points for $f(x)$.

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f''(x) = 36x^2 - 24x - 24 = 0$$

$$3x^2 - 2x - 2 = 0$$

<u>QF</u>:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 24}}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{28}}{6} = \frac{1}{3} \pm \frac{\sqrt{7}}{3} = \frac{1 \pm \sqrt{7}}{3}.$$

$f''(x) = 0$ when $x = \frac{1 \pm \sqrt{7}}{3}$.

These are the inflection points for $f(x)$.

$f''(x) < 0$ on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$

$f''(x) > 0$ on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right) \cup \left(\frac{1+\sqrt{7}}{3}, \infty\right)$.

$f(x)$ is concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$

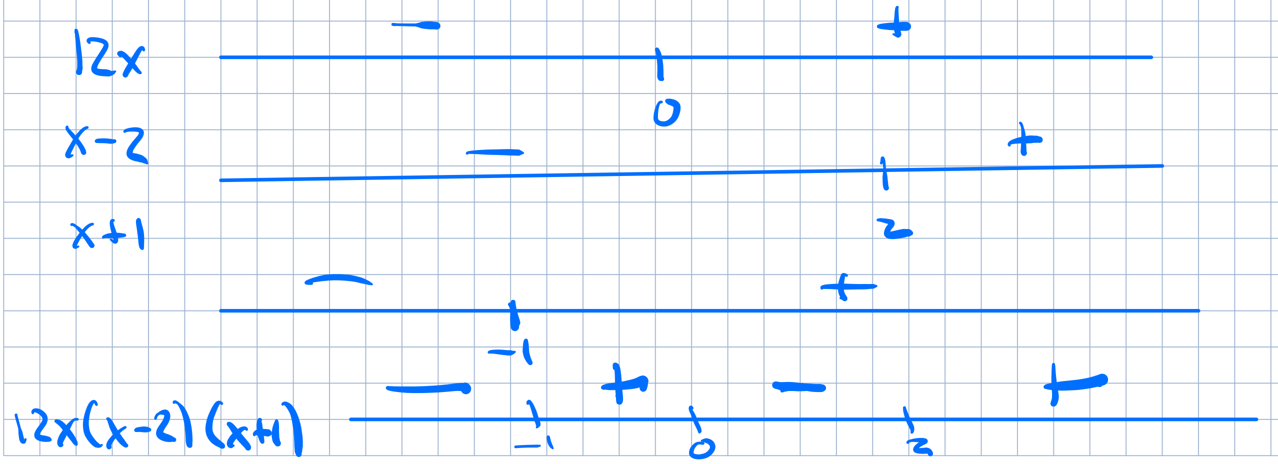
and concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right) \cup \left(\frac{1+\sqrt{7}}{3}, \infty\right)$

Continuing with this example, where does $f(x)$ have a max or a min?

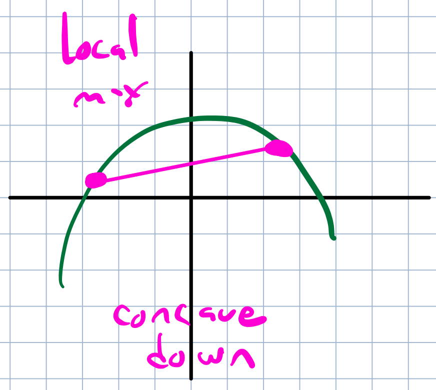
$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1) = 0$$

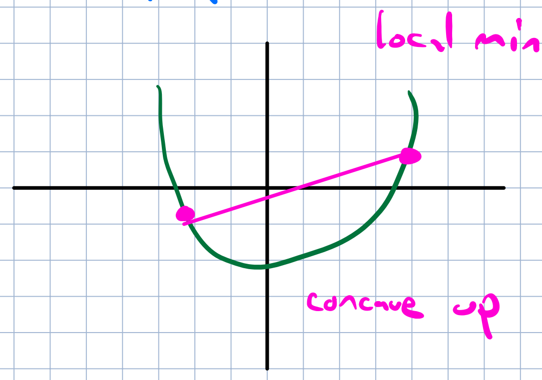
when $x = 0, -1$, or 2

First Derivative Test



↑ local min ↑ local max ↑ local min





Second Derivative Test:

If $f'(x) = 0$ and $f''(x) < 0$, then x is a local max.

If $f'(x) = 0$ and $f''(x) > 0$, then x is a local min.

Returning to our example, we had 3 numbers where $f'(x) = 0$ $x = 0, -1, 2$.

$$f''(x) = 36x^2 - 24x - 24$$

$$f''(0) = 36 \cdot 0^2 - 24 \cdot 0 - 24 = -24 < 0$$

By the 2nd derivative test $x = 0$ is a local max.

$$f''(-1) = 36 \cdot (-1)^2 \cancel{-24(-1)} \cancel{-24} = 36 > 0$$

By the 2nd derivative test, $x = -1$ is a local min.

$$f''(2) = 36\cdot 2^2 - 24\cdot 2 - 24 = 144 - 48 - 24 > 0$$

By the 2nd derivative test, $x = 2$ is a local min.

Ex: Consider $f(x) = x(1-x)^{2/3}$.

Find local maxima and minima and inflection pts.

$$f'(x) \overset{\text{PRODUCT RULE}}{=} x\frac{d}{dx}\left[(1-x)^{2/3}\right] + 1\cdot(1-x)^{2/3}$$

$$= x\cdot\frac{2}{3}(1-x)^{-1/3}\cdot(-1) + (1-x)^{2/3}$$

$$= -\frac{2}{3}x(1-x)^{-1/3} + (1-x)^{2/3}$$

$$= \frac{-2x}{3(1-x)^{1/3}} + (1-x)^{2/3}$$

$$= \frac{-2x}{3(1-x)^{1/3}} + \frac{3-x}{3(1-x)^{1/3}} = \frac{3-3x}{3(1-x)^{1/3}}.$$

$$= \frac{1-x}{(1-x)^{1/3}} = (1-x)^{2/3}.$$

Ex: $f(x) = x^3 - 3x^2 + 4x + 1$.

Find local maxima + minima and inflection points.

$f'(x) = 3x^2 - 6x + 4 = 0$

QF: $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4\cdot 3\cdot 4}}{2\cdot 3} = \dfrac{6 \pm \sqrt{36-48}}{6}$

$f'(x) \neq 0$ for any values of x,

so $f(x)$ has NO local maxima or minima.

$f''(x) = 6x - 6 = 0$ when $x - 1 = 0$

$x = 1$.

Inflection point at $x = 1$.

$f''(x) = 6x - 6 < 0$ when $x < 1$

$6x - 6 > 0$ when $x > 1$, so

$f(x)$ is concave down on $(-\infty, 1)$

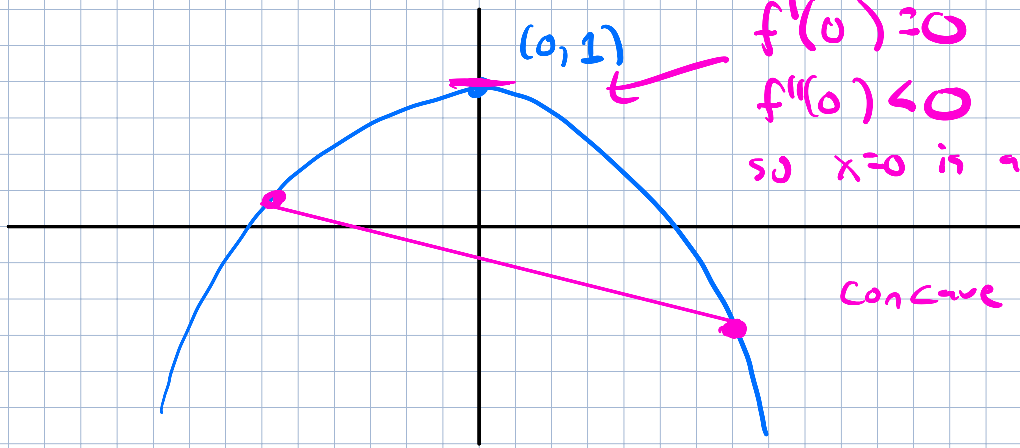
concave up on $(1, \infty)$.

Ex: $f(x) = 1 - x^2$. Find extrema.

$f'(x) = -2x = 0$ when $x = 0$.

$f''(x) = -2 < 0$.

By the second derivative test, $x = 0$ is a local max.



$f'(0) = 0$

$f''(0) < 0$

so $x = 0$ is a local max.

concave down