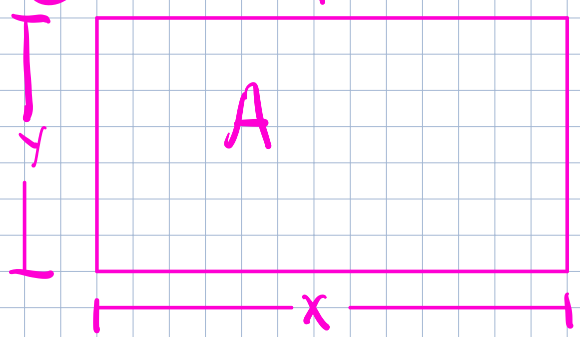
Exam 3 <u>IN CLASS</u> Monday, April 11

Final Project due on Canvas Monday, April 18

Optimization

A farmer has 1600 feet of fence and wants to use it to build a rectangular enclosure with the maximum area. What dimensions should the rectangular enclosure have?

(1) Draw a picture and label the relevant variables



(2) What are we given? What are we trying to maximize/minimize?

Given: $2x + 2y = 1600$

We want to maximize A.

(3) Write the function we're trying to maximize/minimize as a function of a <u>single</u> variable.

$$A = xy$$

$$2x + 2y = 1600 \qquad x + y = 800$$

$$y = 800 - x$$

$$A = x(800 - x) = 800x - x^2$$

Now, the problem becomes one of finding maxima or minima just like we've been doing for the past $1\frac{1}{2}$ weeks.

(4) What are the endpoints?

$[0, 800]$.

(5) Take the derivative of the function you want to optimize.

$$\frac{dA}{dx} = \frac{d}{dx}\left[800x - x^2\right]$$

$$= 800 - 2x = 0$$

when $2x = 800$

$$x = 400.$$

(6) Which of these is the max?

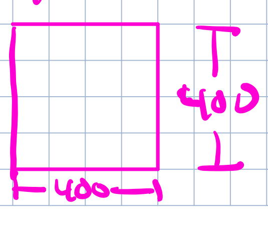
$$A(0) = 800 \cdot 0 - 0^2 = 0$$

$$A(800) = 800 \cdot 800 - 800^2 = 0$$

$$A(400) = 800 \cdot 400 - 400^2 = 320{,}000 - 160{,}000$$
$$= 160{,}000$$

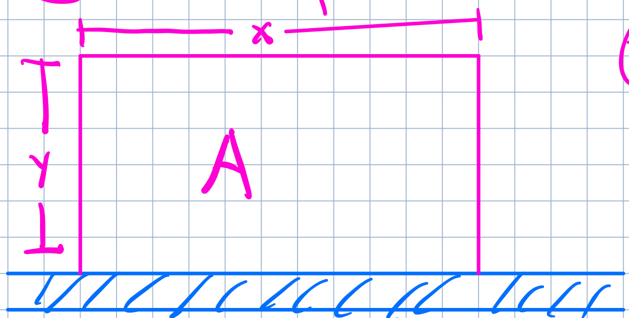
The area is maximized when $x = 400$ ft.

$$y = 800 - x = 800 - 400 = 400$$



Ex: Suppose now that there is a straight stream running through the farmer's land. They want to build the rectangular enclosure using the stream as one of the sides. Now, what dimensions would maximize the area?

① Draw a picture + label the variables



② What are we trying to optimize? What are we given?

We want to maximize A.

Given: $x + 2y = 1600$

③ Write the function as a function of a single variable.

$$A = xy$$

$$x + 2y = 1600 \qquad 2y = 1600 - x$$

$$y = 800 - \tfrac{1}{2}x$$

$$A = x\left(800 - \tfrac{1}{2}x\right) = 800x - \tfrac{1}{2}x^2$$

④ Where are the endpoints? $[0, 1600]$

⑤ Where is the derivative zero? Where is the derivative undefined?

$$\frac{dA}{dx} = \frac{d}{dx}\left[800x - \tfrac{1}{2}x^2\right] = 800 - x = 0$$

$800 = x$.

⑥ Which of these is the max?

$A(0) = 800 \cdot 0 - \frac{1}{2} \cdot 0^2 = 0$

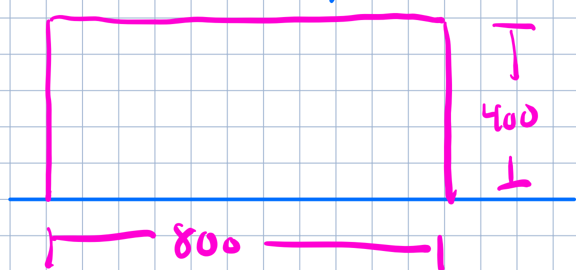
$A(1600) = 800 \cdot 1600 - \frac{1}{2} \cdot 1600^2 = 0$

$A(800) = 800 \cdot 800 - \frac{1}{2} \cdot 800^2 = 640{,}000 - 320{,}000$

$= 320{,}000$.

The max occurs when $x = 800$ ft.

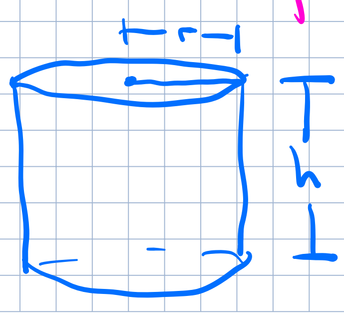
$y = 800 - \frac{1}{2}x = 800 - \frac{1}{2} \cdot 800 = 400$ ft.



Ex: I want to build a cylindrical can with volume 360 cm^3 with the minimum possible surface area.

What dimensions should it have?

① Draw a picture

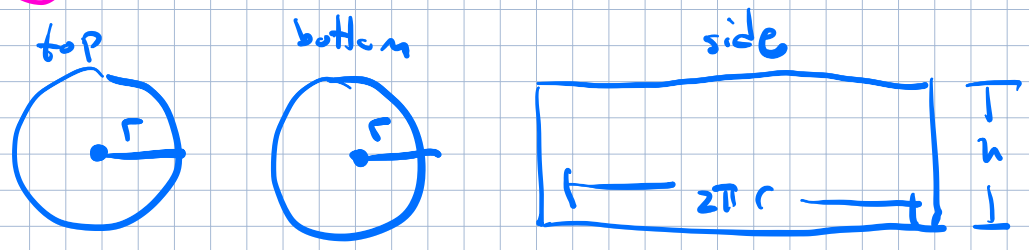


② What are we trying to minimize? What are we given?

We want to minimize A: surface area.

Given: $V = 360\ cm^3$.

③ Write A as a function of a single variable.



$\pi r^2 \qquad \pi r^2 \qquad 2\pi r h$

$$A = 2\pi r^2 + 2\pi r h = \cancel{2\pi r(r+h)}$$

$$V = 360 \text{ cm}^3.$$

$$360 = V = \pi r^2 \cdot h \qquad h = \frac{360}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r \cdot \frac{360}{\pi r^2}$$

$$= 2\pi r^2 + \frac{720}{r}.$$

④ What are the endpoints? $[0, \infty)$

⑤ Where is $\frac{dA}{dr} = 0$? Where is $\frac{dA}{dr}$ undefined?

$$\frac{dA}{dr} = \frac{d}{dr}\left[2\pi r^2 + \frac{720}{r}\right] = 4\pi r + \left(-\frac{720}{r^2}\right)$$

$$= 4\pi r - \frac{720}{r^2}$$

undefined when $r=0$.

$$4\pi r - \frac{720}{r^2} = 0 \quad \text{when} \quad 4\pi r^3 - 720 = 0$$

$$4\pi r^3 = 720$$

$$r^3 = \frac{180}{\pi}. \qquad r = \sqrt[3]{\frac{180}{\pi}} \text{ cm.}$$

Is this a max or a min? 2nd derivative test.

$$\frac{d}{dr}\left[4\pi r - \frac{720}{r^2}\right] = 4\pi + \frac{1440}{r^3}.$$

$$4\pi + \frac{1440}{\left(\sqrt[3]{\frac{180}{\pi}}\right)^3} = 4\pi + \frac{1440}{\frac{180}{\pi}} > 0$$

by the 2nd derivative test, $r = \sqrt[3]{\frac{180}{\pi}}$ is a local min.

$$h = \frac{360}{\pi r^2} = \frac{360}{\pi\left(\sqrt[3]{\frac{180}{\pi}}\right)^2}$$