Exam 3 IN CLASS Monday, April 11

the topic of today's lecture WILL be on the exam. Wednesday's will NOT.

Final Project due on Canvas Monday, April 18

L'Hôpital's Rule

Useful for calculating certain kinds of limits.

Ex: $\lim_{x \to 0} \frac{e^x - 1}{x}$.

L'Hôpital's Rule: Suppose that f and g are differentiable functions and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Ex: Compute $\lim_{x \to 0} \frac{e^x - 1}{x}$.

$$\lim_{x \to 0} [e^x - 1] = e^0 - 1 = 1 - 1 = 0$$

$$\lim_{x \to 0} [x] = 0$$

Since both the top and the bottom are going to 0, we can use L'Hôpital's Rule

$$\lim_{x\to 0} \frac{e^x - 1}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{d}{dx}[e^x - 1]}{\frac{d}{dx}[x]} = \lim_{x\to 0} \frac{e^x}{1}$$

$$= \frac{e^0}{1} = 1.$$

Ex: Compute $\displaystyle\lim_{x\to 0} \frac{1-\cos^2(x)}{\sin(x)}$

$$\lim_{x\to 0} [1-\cos^2(x)] = 1-\cos^2(0) = 1-1^2 = 0$$

$$\lim_{x\to 0} [\sin(x)] = \sin(0) = 0$$

Both the top and the bottom are going to 0, so we can use L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1-\cos^2(x)}{\sin(x)} = \lim_{x\to 0} \frac{\frac{d}{dx}[1-\cos^2(x)]}{\frac{d}{dx}[\sin(x)]}$$

$$= \lim_{x\to 0} \frac{+2\cos(x)\sin(x)}{\cos(x)} = \lim_{x\to 0} \frac{2\sin(x)}{1}$$

$$= 2\sin(0) = 2\cdot 0 = 0.$$

Ex: Compute $\displaystyle\lim_{x\to \infty} \frac{\ln(x)}{x}$.

$$\lim_{x\to\infty} \ln(x) = \infty \qquad \lim_{x\to\infty} x = \infty.$$

You can also use L'Hôpital's Rule in the situation where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x]} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

Ex: $$\lim_{x\to\infty} \frac{x^3-3x+1}{3x^3-2x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}[x^3-3x+1]}{\frac{d}{dx}[3x^3-2x^2]}$$

$$= \lim_{x\to\infty} \frac{3x^2-3}{9x^2-4x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[3x^2-3]}{\frac{d}{dx}[9x^2-4x]}$$

$$= \lim_{x\to\infty} \frac{6x}{18x-4} = \lim_{x\to\infty} \frac{\frac{d}{dx}[6x]}{\frac{d}{dx}[18x-4]}$$

$$= \lim_{x\to\infty} \frac{6}{18} = \frac{1}{3}.$$

Ex: Compute $\lim_{x\to 0^+} x \cdot \ln(x)$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

$$\begin{cases} \lim_{x\to 0^+} \ln(x) = -\infty \\ \lim_{x\to 0^+} \frac{1}{x} = \infty \end{cases}$$

So we can apply L'Hôpital's Rule

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{-\frac{1}{x}} = \lim_{x\to 0^+} -x$$

$$= 0$$

Ex: Compute $\lim_{x\to(\frac{\pi}{2})^-} [\tan(x) - \sec(x)]$.

Indeterminate form $\infty - \infty$.

$$= \lim_{x\to(\frac{\pi}{2})^-} \left[\frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)}\right] = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sin(x) - 1}{\cos(x)}$$

top + bottom are both going to 0, so we can use L'Hôpital's Rule

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{d}{dx}[\sin(x) - 1]}{\frac{d}{dx}[\cos(x)]} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\cos(x)}{-\sin(x)}$$

$$= \frac{\cos(\pi/2)}{-\sin(\pi/2)} = \frac{0}{-1} = 0.$$

Ex: Compute $\lim_{x\to\infty} \left[x - \sqrt{x^2 + x}\right]$

$$= \lim_{x\to\infty} \left[x \cdot \left(1 - \sqrt{1 + \frac{1}{x}}\right)\right]$$

$$= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}}$$

both the top and the bottom are going to 0, so we can use L'Hôpital

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\left[1-\sqrt{1+\frac{1}{x}}\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]} = \lim_{x\to\infty} \frac{-\frac{1}{2\sqrt{1+\frac{1}{x}}}\cdot\frac{d}{dx}\left[1+\frac{1}{x}\right]}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{2\sqrt{1+\frac{1}{x}}}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} -\frac{1}{2\sqrt{1+\frac{1}{x}}}$$

$$= -\frac{1}{2\sqrt{1+0}} = -\frac{1}{2}$$

Ex: Compute $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)}$

$$= \lim_{x\to 0^+} e^{x\cdot\ln(x)}$$

Let's first compute $\lim_{x\to 0^+} x\cdot\ln(x)$.

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{1}{-\frac{1}{x}} = \lim_{x\to 0^+} -x = 0.$$

So $\lim_{x \to 0^+} e^{x \cdot \ln(x)} = e^{\lim_{x \to 0^+} x \cdot \ln(x)} = e^0$

$= 1.$