Exam 3 Monday April 11 IN CLASS

What's on the Exam?

Derivatives

- derivatives of trig functions
- derivatives of exponentials
- derivatives of logs / logarithmic differentiation

Applications of Derivatives

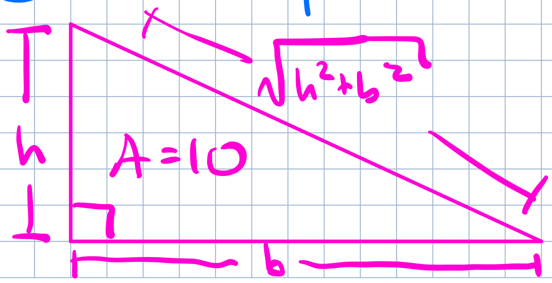
Graphing
- maxima + minima
- increasing + decreasing
- concavity + inflection points

- Mean Value Theorem
- Optimization ←
- L'Hôpital's Rule

Optimization

Ex: Of all right triangles with area 10, which one has the smallest perimeter?

① Draw a picture



② What are we trying to optimize? what are we given?

Minimize perimeter P.

Given: $A=10$.

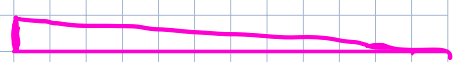
(3) Write P as a function of a <u>single</u> variable.

$$P = b + h + \sqrt{h^2 + b^2}$$

$$10 = A = \frac{1}{2} \cdot b \cdot h \qquad 20 = b \cdot h \qquad \frac{20}{b} = h$$

$$P = b + \frac{20}{b} + \sqrt{\left(\frac{20}{b}\right)^2 + b^2}$$

$$= b + \frac{20}{b} + \sqrt{\frac{400}{b^2} + b^2}$$

(4) What are the endpoints? 

b is in the interval $(0, \infty)$

P is not guaranteed to have a minimum!

(5) Take the derivative

$$\frac{dP}{db} = \frac{d}{db}\left[b + \frac{20}{b} + \sqrt{\frac{400}{b^2} + b^2}\right]$$

$$= 1 - \frac{20}{b^2} + \frac{\frac{d}{db}\left[\frac{400}{b^2} + b^2\right]}{2\sqrt{\frac{400}{b^2} + b^2}}$$

$$= 1 - \frac{20}{b^2} + \frac{-\frac{800}{b^3} + 2b}{2\sqrt{\frac{400}{b^2} + b^2}} = 0$$

$$\frac{\frac{-800}{b^3}+2b}{2\sqrt{\frac{400}{b^2}+b^2}} = \frac{20}{b^2}-1$$

$$-\frac{800}{b^3}+2b = \frac{40\sqrt{\frac{400}{b^2}+b^2}}{b^2} - 2\sqrt{\frac{400}{b^2}+b^2}$$

$$-800+2b^4 = 40b\sqrt{\frac{400}{b^2}+b^2} - 2b^3\sqrt{\frac{400}{b^2}+b^2}$$

Solve for b.

You will get $b=\sqrt{20}=2\sqrt{5}$.

(5) How do we know this is a local min?

Second derivative test.

$$\frac{d^2P}{db^2} = \frac{d}{db}\left[1-\frac{20}{b^2}+\frac{\frac{-800}{b^3}+2b}{2\sqrt{\frac{400}{b^2}+b^2}}\right]$$

$$= \frac{40}{b^3}+\frac{d}{db}\left[\frac{-\frac{400}{b^3}+b}{\sqrt{\frac{400}{b^2}+b^2}}\right]$$

$$= \frac{40}{b^3} + \frac{\sqrt{\frac{400}{b^2}+b^2} \cdot \frac{d}{db}\left[-\frac{400}{b^3}+b\right] - \left[-\frac{400}{b^3}+b\right] \cdot \frac{d}{db}\left(\sqrt{\frac{400}{b^2}+b^2}\right)}{\frac{400}{b^2}+b^2}$$

$$= \frac{40}{b^3} + \frac{\sqrt{\frac{400}{b^2}+b^2}\left(\frac{1200}{b^4}+1\right) + \left(\frac{400}{b^3}-b\right)\left(\frac{-\frac{400}{b^3}+b}{\sqrt{\frac{400}{b^2}+b^2}}\right)}{\frac{400}{b^2}+b^2}$$

$$p''(2\sqrt{5}) = \underline{\qquad\qquad\qquad} > 0.$$

P is concave up, so $b = 2\sqrt{5}$ is a local min.

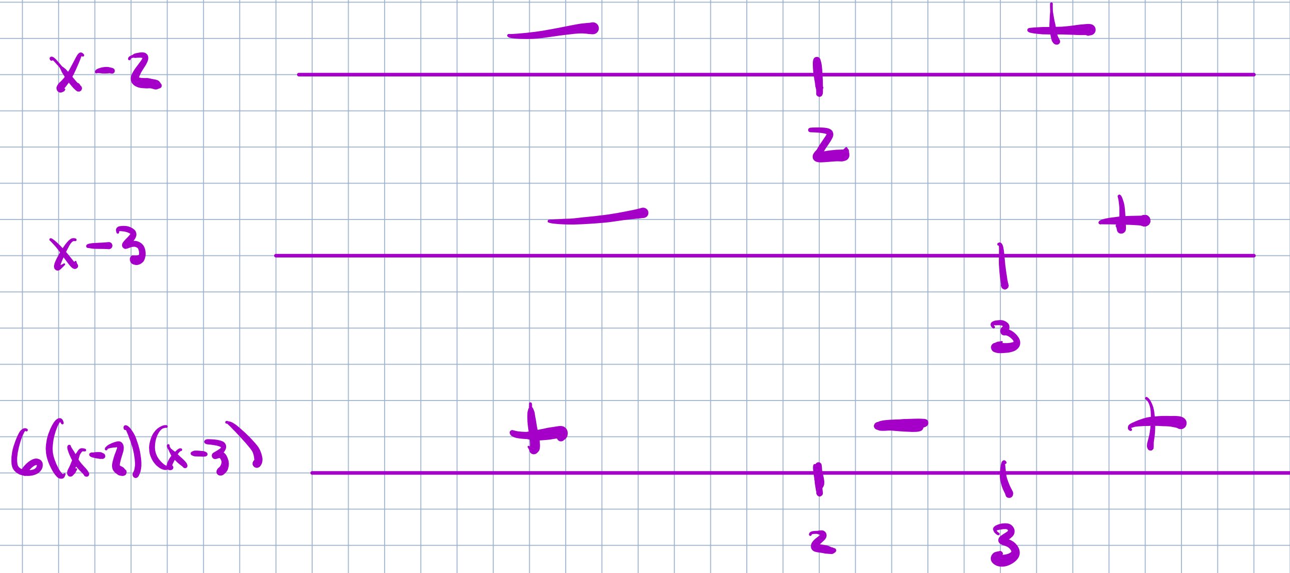
Graphing

Ex: Consider $f(x) = 2x^3 - 15x^2 + 36x + 7$.

Where are the local maxima / minima?
Where is f increasing / decreasing?
Where are the inflection points?
Where is f concave up / concave down?

$$f'(x) = 6x^2 - 30x + 36$$
$$= 6(x^2 - 5x + 6) = 6(x-2)(x-3)$$

So $f'(x) = 0$ when $x = 2, 3$.



$f'(x) > 0$ when $x < 2$ or $x > 3$

$f'(x) < 0$ when $2 < x < 3$.

$f(x)$ is increasing when $x < 2$ or $x > 3$

$f(x)$ is decreasing when $2 < x < 3$.

So $x = 2$ is a local max.

So $x = 3$ is a local min.

$$f''(x) = \frac{d}{dx}\left[6x^2 - 30x + 36\right]$$

$$= 12x - 30 = 12\left(x - \frac{5}{2}\right).$$

$f''(x) = 0$ when $x = \frac{5}{2}$.

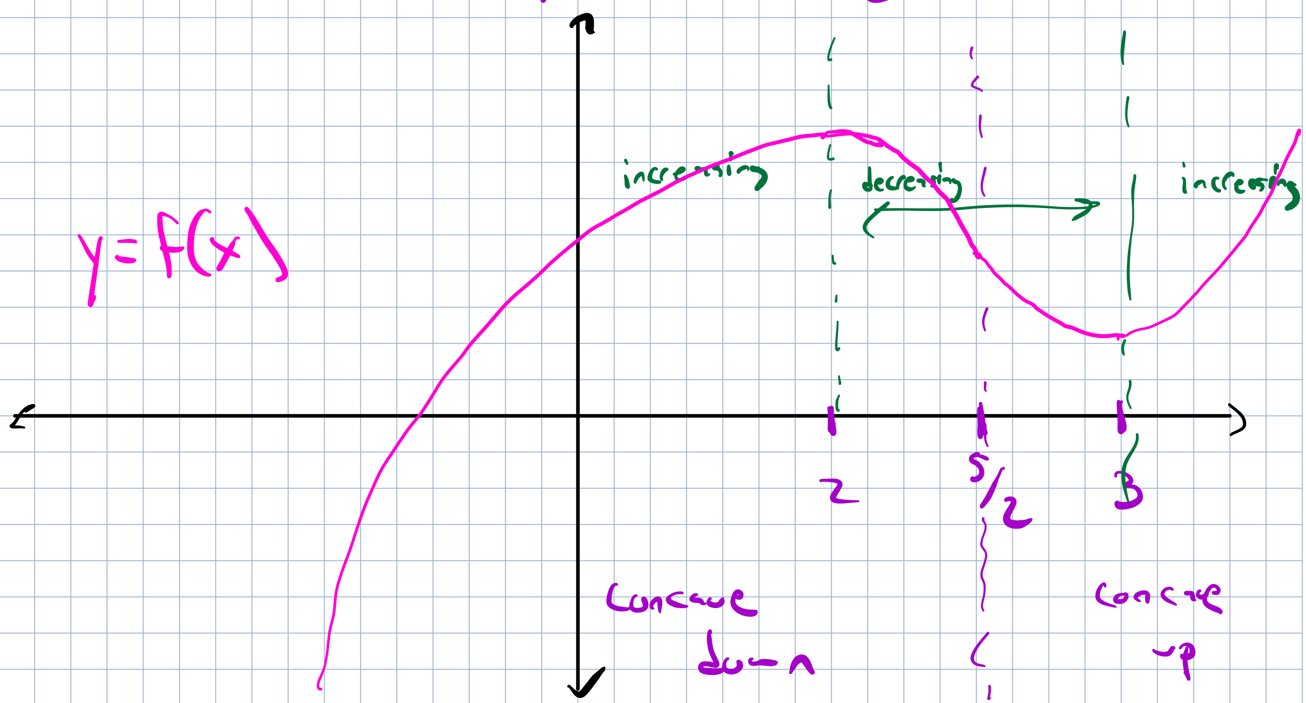
$x = \frac{5}{2}$ is an inflection point.

$f''(x) < 0$ when $x < \frac{5}{2}$

$f''(x) > 0$ when $x > \frac{5}{2}$.

$f(x)$ is concave down when $x < \frac{5}{2}$

$f(x)$ is concave up when $x > \frac{5}{2}$.

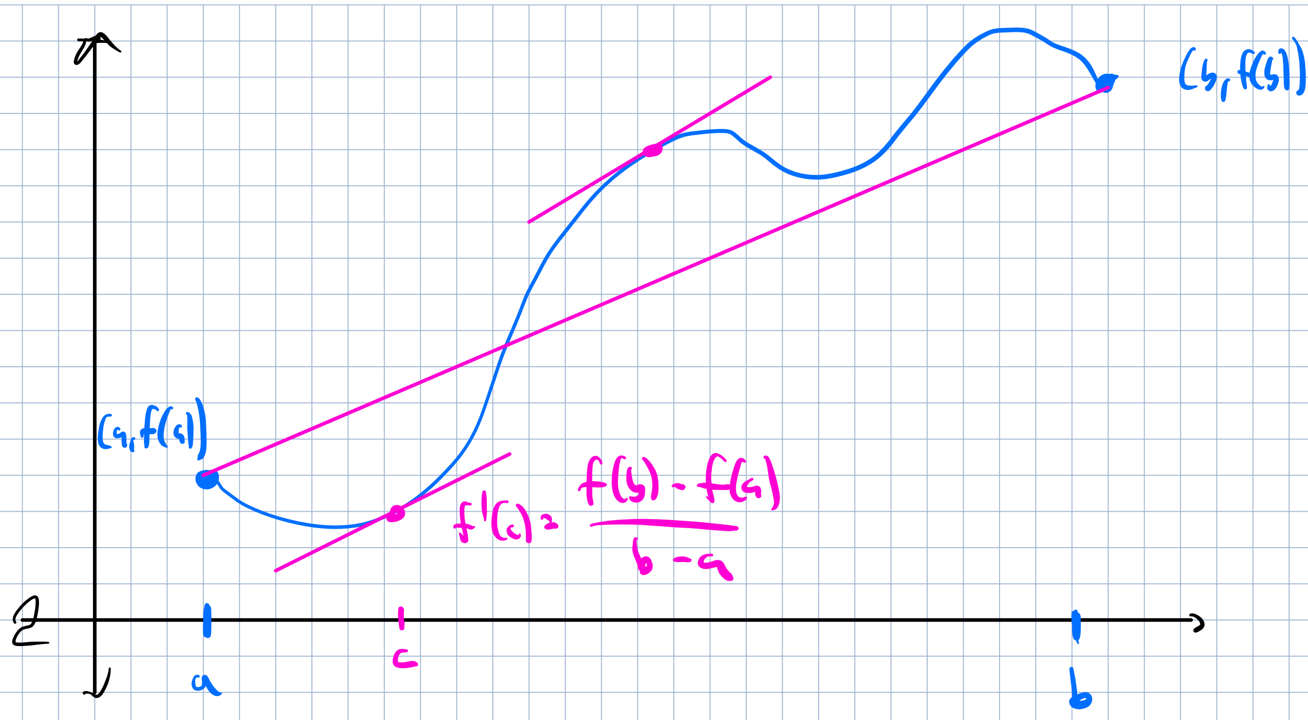


Mean Value Theorem

Let $f(x)$ be a differentiable function on $[a,b]$.

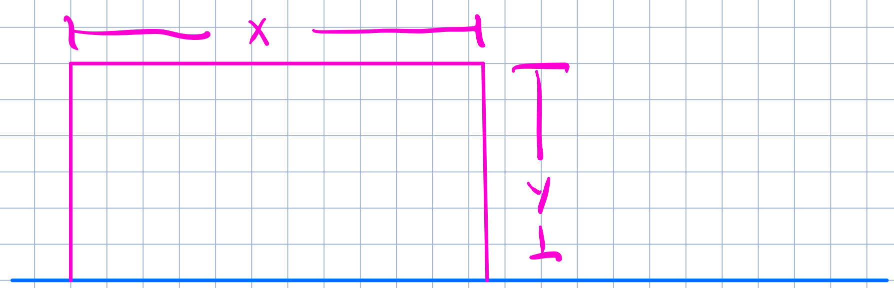
Then there is a number c in $[a,b]$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Another Optimization Problem



perimeter of the fence = 100 yds

Maximize area

$$A = xy \qquad x + 2y = 100$$

$$x = 100 - 2y$$

$$= (100 - 2y)y = 100y - 2y^2$$

$$\frac{dA}{dy} = \frac{d}{dy}[100y - 2y^2] = 100 - 4y = 0$$

$100 = 4y \qquad 25 = y. \qquad x = 100 - 2y$

$= 100 - 2 \cdot 25$

$= 50.$

To see that this is a local max, let's use the 2nd derivative test.

$$\frac{d^2A}{dy^2} = \frac{d}{dy}[100 - 4y] = -4 < 0$$

A is always concave down.

So $y = 25$ is a local max.