Exam 1 IN CLASS Wednesday, Feb 2nd

Limits of Recursively Defined Sequences

Ex: $a_{n+1} = \frac{1}{2} \cdot a_n$, $a_0 = 48$

$a_0 = 48$

$a_1 = \frac{1}{2} \cdot a_0 = \frac{1}{2} \cdot 48 = 24$

$a_2 = \frac{1}{2} \cdot a_1 = \frac{1}{2} \cdot 24 = 12$

$a_3 = \frac{1}{2} \cdot a_2 = \frac{1}{2} \cdot 12 = 6$

$a_4 = \frac{1}{2} \cdot a_3 = \frac{1}{2} \cdot 6 = 3$

$a_5 = \frac{1}{2} \cdot a_4 = \frac{1}{2} \cdot 3 = \frac{3}{2}$

$48, 24, 12, 6, 3, \frac{3}{2}, \ldots$

$a_n = 48 \cdot \left(\frac{1}{2}\right)^n$.

$a_{1000} = 48 \cdot \left(\frac{1}{2}\right)^{1000} \approx 0.00001$

As n gets large, a_n get closer to 0.

$\lim_{n \to \infty} a_n = 0$.

If $a_{n+1} = \frac{1}{2} \cdot a_n$, $\underline{\underline{a_0 = 0}}$, then

$a_0 = 0$

$a_1 = \frac{1}{2} \cdot a_0 = \frac{1}{2} \cdot 0 = 0$.

$a_2 = \frac{1}{2} \cdot a_1 = \frac{1}{2} \cdot 0 = 0$

$a_3 = \frac{1}{2} \cdot a_2 = \frac{1}{2} \cdot 0 = 0$

$0, 0, 0, 0, 0, 0, \ldots$

__Def__: A number a is called a __fixed point__ for a recursion if, when $a_n = a$, then $a_{n+1} = a$.

__Ex__: In the example above $a = 0$ is a fixed point.

$a_{n+1} = \frac{1}{2} \cdot a_n$

To find the fixed points, plug a in for a_n and a_{n+1}, and then solve.

$a = \frac{1}{2} a$. $\quad \frac{1}{2} a = 0 \quad \boxed{a = 0.}$

__Fact__: If a_n is a recursively defined sequence, and $\lim_{n \to \infty} a_n = a$, then a is a fixed point for the recursion.

In the previous example, the only fixed

point was $a=0$, so if the sequence has a limit, that limit has to be $a=0$.

Ex: $a_{n+1} = 2 \cdot a_n$, $a_0 = 1$

$a_0 = 1$

$a_1 = 2 \cdot a_0 = 2 \cdot 1 = 2$

$a_2 = 2 \cdot a_1 = 2 \cdot 2 = 4$

$a_3 = 2 \cdot a_2 = 2 \cdot 4 = 8$

$a_n = 2^n$

1, 2, 4, 8, 16, 32, 64, 128, 256, ...

To find the fixed points for this recursion, set $a_n = a_{n+1} = a$ and solve for a.

$a_{n+1} = 2 \cdot a_n$

$a = 2 \cdot a$ $\boxed{0 = a}$ ← fixed point

BUT $\lim_{n \to \infty} a_n$ does not exist.

What's the difference between these two?

In $a_{n+1} = \frac{1}{2} \cdot a_n$, if you plug in a value for a_n that is very close but not equal to 0, then a_{n+1} is even closer to 0. $\leftarrow$ $\boxed{\text{Locally stable fixed point}}$

Ex: $a_n = \frac{1}{1000}$, then $a_{n+1} = \frac{1}{2000}$

In $a_{n+1} = 2 \cdot a_n$, if you plug in a value for a_n that is very close but not equal to 0, then a_{n+1} is farther away from 0 than a_n is.

$\boxed{\text{Locally unstable fixed point}}$

Ex: $a_n = \frac{1}{1000}$, $a_{n+1} = \frac{2}{1000} = \frac{1}{500}$

If $\lim\limits_{n\to\infty} a_n = a$, then a is a locally stable fixed point for the recursion.

Ex: $a_{n+1} = \frac{-2}{a_n+3}$ $\quad a_0 = 0$.

Find $\lim\limits_{n\to\infty} a_n$ if it exists.

First, find the fixed points.

$$a = \frac{-2}{a+3} \qquad a(a+3) = -2$$

$$a^2 + 3a = -2 \qquad a^2 + 3a + 2 = 0$$

$$a = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-3 \pm \sqrt{9-8}}{2}$$

$$= \frac{-3 \pm 1}{2} = -1, -2$$

$a^2+3a+2 = (a+2)(a+1) = 0$

Recursion has 2 fixed points: $a = -1, -2$.

Which of these is the limit of our sequence?

$$a_{n+1} = \frac{-2}{a_n+3} \qquad a_0 = 0$$

$a_0 = 0$

$$a_1 = \frac{-2}{0+3} = -\frac{2}{3}$$

$$a_2 = \frac{-2}{-\frac{2}{3}+3} = \frac{-2}{\frac{7}{3}} = \frac{-6}{7}$$

$$a_3 = \frac{-2}{-\frac{6}{7}+3} = \frac{-2}{\frac{15}{7}} = \frac{-14}{15}$$

These are getting closer to -1,

so $\lim_{n\to\infty} a_n = -1$.

Another approach: which of the 2 fixed points is locally stable?

For $a = -1$, plug in $a_n = -1.001$

$$a_{n+1} = \frac{-2}{-1.001+3} = -1.00050025$$

This is closer to -1 than -1.001 is.

So $a=-1$ is a locally stable fixed point.

For $a=-2$, plug in $a_n = -2.001$

$$a_{n+1} = \frac{-2}{-2.001+3} = -2.002002002$$

This is further from -2 than -2.001 is.

So $a=-2$ is a locally unstable fixed point.

So $\lim_{n\to\infty} a_n = -1$ b/c -1 is the only locally stable fixed point.

<u>Ex</u>: $a_{n+1} = \sqrt{a_n}$, $a_0 = 2$.

Find $\lim_{n\to\infty} a_n$, if it exists.

Find the fixed points.

$a = \sqrt{a}$ $\qquad a^2 = a$

$a^2 - a = 0$ $\qquad a(a-1) = 0$

Fixed points are $a=0$, $a=1$.

$a_0 = 2$

$a_1 = \sqrt{2} = 1.414\ldots$

$a_2 = \sqrt{a_1} = \sqrt{1.414\ldots} = 1.1892\ldots$

$a_3 = \sqrt{a_2} = \sqrt{1.1892\ldots} = 1.0905\ldots$

$\lim_{n \to \infty} a_n = 1.$

Which point is locally stable?

For $a = 0$, plug in $a_n = \frac{1}{100}$

$a_{n+1} = \sqrt{\frac{1}{100}} = \frac{1}{10}$

This is further from 0 than $\frac{1}{100}$,

so $a = 0$ is a locally unstable fixed point.

For $a = 1$, plug in $a_n = 1.001$

$a_{n+1} = \sqrt{1.001} = 1.000499\ldots$

This is closer to 1 than 1.001 is,

so $a = 1$ is a locally stable fixed point.

$\lim_{n \to \infty} a_n = 1.$

N factorial

$N! = N \cdot (N-1) \cdot (N-2)(N-3) \cdots 1$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$