The Bernoullis and the Harmonic Series

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After receiving a B.S. in mathematics from the University of Pittsburgh in 1969, William Dunham moved westward to complete a 1974 Ph.D. at the Ohio State University, where he did his thesis in general topology under Professor Norman Levine. Since graduate school, Dr. Dunham has taught mathematics at Hanover College, an institution where faculty must be generalists and where the liberal arts are taken seriously. In that environment, his interests shifted toward the history of mathematics. Professor Dunham received a grant, in 1983, from the Lilly Endowment, Inc., to pursue these historical interests, and it was while engaged in such pursuits that he stumbled upon the 300-year old mathematical morsel described in this paper.

Any introduction to the topic of infinite series soon must address that first great counterexample of a divergent series whose general term goes to zero—the harmonic series $\sum_{k=1}^{\infty} 1/k$. Modern texts employ a standard argument, traceable back to the great 14th Century Frenchman Nicole Oresme (see [3], p. 92), which establishes divergence by grouping the partial sums:

$$1+\frac{1}{2}>\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$$

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)>\frac{2}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)=\frac{3}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)>\frac{3}{2}+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=\frac{4}{2},$$

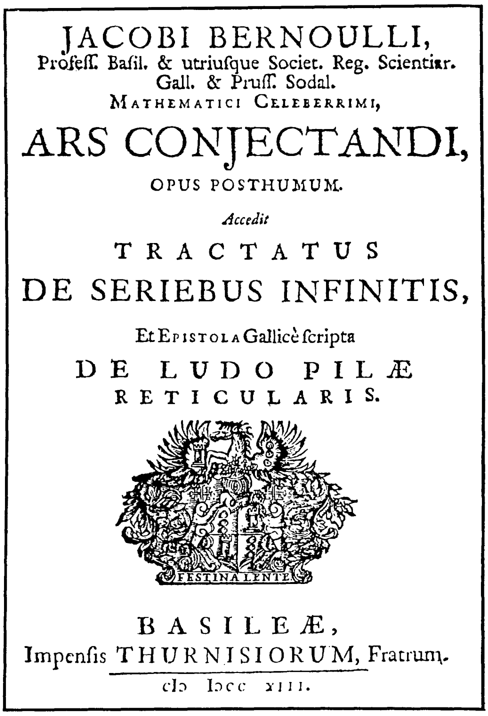
and in general

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}>\frac{n+1}{2},$$

from which it follows that the partial sums grow arbitrarily large as n goes to infinity.

It is possible that seasoned mathematicians tend to forget how surprising this phenomenon appears to the uninitiated student—that, by adding ever more negligible terms, we nonetheless reach a sum greater than any preassigned quantity. Historian of mathematics Morris Kline ([5], p. 443) reminds us that this feature of the harmonic series seemed troubling, if not pathological, when first discovered.

So unusual a series could not help but attract the interest of the preeminent mathematical family of the 17th Century, the Bernoullis. Indeed, in his 1689 treatise "Tractatus de Seriebus Infinitis," Jakob Bernoulli provided an entirely different, yet equally ingenious proof of the divergence of the harmonic series. In "Tractatus," which is now most readily found as an appendix to his posthumous 1713 masterpiece *Ars Conjectandi*, Jakob generously attributed the proof to his brother ("Id primus deprehendit Frater"), the reference being to his full-time sibling and part-time rival Johann. While this "Bernoullian" argument is sketched in such mathematics history texts as Kline ([5], p. 444) and Struik ([6], p. 321), it is little enough known to warrant a quick reexamination.



JACOBI BERNOULLI,
Profeſſ. Baſil. & utriuſque Societ. Reg. Scientiar. Gall. & Pruſſ. Sodal.
MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,

OPUS POSTHUMUM.

Accedit

TRACTATUS
DE SERIEBUS INFINITIS,

Et EPISTOLA Gallicè ſcripta
DE LUDO PILÆ RETICULARIS.

BASILEÆ,
Impenſis THURNISIORUM, Fratrum.
cIↃ IↃCC XIII.

Courtesy of the Lilly Library, Indiana University, Bloomington, IN

The proof rested, quite unexpectedly, upon the *convergent* series

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}.$$

The modern reader can easily establish, via mathematical induction, that

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1},$$

and then let n go to infinity to conclude that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=1.$$

Jakob Bernoulli, however, approached the problem quite differently. In Section XV of *Tractatus*, he considered the infinite series

$$N = \frac{a}{c} + \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \cdots,$$

then introduced

$$P = N - \frac{a}{c} = \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \frac{a}{5c} + \cdots,$$

and subtracted termwise to get

$$\begin{aligned} \frac{a}{c} = N - P &= \left(\frac{a}{c} - \frac{a}{2c}\right) + \left(\frac{a}{2c} - \frac{a}{3c}\right) + \left(\frac{a}{3c} - \frac{a}{4c}\right) + \cdots \\ &= \frac{a}{2c} + \frac{a}{6c} + \frac{a}{12c} + \frac{a}{20c} + \cdots. \end{aligned} \tag{1}$$

Thus, for $a = c$, he concluded that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = \frac{1}{1} = 1. \tag{2}$$

Unfortunately, Bernoulli's "proof" required the subtraction of two divergent series, N and P. To his credit, Bernoulli recognized the inherent dangers in his argument, and he advised that this procedure must not be used without caution ("non sine cautela"). To illustrate his point, he applied the previous reasoning to the series

$$S = \frac{2a}{c} + \frac{3a}{2c} + \frac{4a}{3c} + \cdots$$

and

$$T = S - \frac{2a}{c} = \frac{3a}{2c} + \frac{4a}{3c} + \frac{5a}{4c} + \cdots.$$

Upon subtracting termwise, he got

$$\frac{2a}{c} = S - T = \frac{a}{2c} + \frac{a}{6c} + \frac{a}{12c} + \frac{a}{20c} + \cdots, \tag{3}$$

which provided a clear contradiction to (1).

Bernoulli analyzed and resolved this contradiction as follows: the derivation of (1) was valid since the "last" term of series N is zero (that is, $\lim_{k \to \infty} a/(kc) = 0$), whereas the parallel derivation of (3) was invalid since the "last" term of series S is non-zero (because $\lim_{k \to \infty} (k+1)a/(kc) = a/c \neq 0$). In modern terms, he had correctly recognized that, regardless of the convergence or divergence of the series $\sum_{k=1}^{\infty} x_k$, the new series $\sum_{k=1}^{\infty} (x_k - x_{k+1})$ converges to x_1 *provided* $\lim_{k \to \infty} x_k = 0$. Thus, he not only explained the need for "caution" in his earlier discussion but also exhibited a fairly penetrating insight, by the standards of his day, into the general convergence/divergence issue.

Having thus established (2) to his satisfaction, Jakob addressed the harmonic series itself. Using his brother's analysis of the harmonic series, he proclaimed in Section XVI of *Tractatus*:

XVI. *Summa seriei infinitæ harmonicè progressionalium*, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ *&c. est infinita.*

He began the argument that "the sum of the infinite harmonic series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\text{ etc.}$$

is infinite" by introducing

$$A=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots,$$

which "transformed into fractions whose numerators are 1, 2, 3, 4 etc" becomes

$$\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}+\cdots.$$

Using (2), Jakob next evaluated:

$$\begin{aligned}
C&=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=1\\
D&=\phantom{\frac{1}{2}+{}}\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=C-\frac{1}{2}=1-\frac{1}{2}=\frac{1}{2}\\
E&=\phantom{\frac{1}{2}+\frac{1}{6}+{}}\frac{1}{12}+\frac{1}{20}+\cdots=D-\frac{1}{6}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}\\
F&=\phantom{\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+{}}\frac{1}{20}+\cdots=E-\frac{1}{12}=\frac{1}{3}-\frac{1}{12}=\frac{1}{4}\\
&\vdots\qquad\qquad\vdots\qquad\vdots\qquad\qquad\vdots
\end{aligned}$$

By adding this array columnwise, and again implicitly assuming that termwise addition of infinite series is permissible, he arrived at

$$\begin{aligned}
C+D+E+F+\cdots&=\frac{1}{2}+\left(\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{12}+\frac{1}{12}+\frac{1}{12}\right)+\cdots\\
&=\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\cdots\\
&=A.
\end{aligned}$$

On the other hand, upon separately summing the terms forming the extreme left and the extreme right of the arrayed equations above, he got

$$C+D+E+F+\cdots=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=1+A.$$

Hence, $A=1+A$. In Jakob's words, "The whole" equals "the part"—that is, the harmonic series $1+A$ equals its part A—which is impossible for a finite quantity. From this, he concluded that $1+A$ is infinite.

XVI. *Summa seriei infinitae harmonicè progressionalium*, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ *&c. est infinita.*

Id primus deprehendit Frater: inventa namque per praeced. summa seriei $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c. visurus porrò, quid emergeret ex ista serie, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}$, &c. si resolveretur methodo Prop. XIV. collegit propositionis veritatem ex absurditate manifesta, quae sequeretur, si summa seriei harmonicae finita statueretur. Animadvertit enim,

Seriem A, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$, &c. ∞ (fractionibus singulis in alias, quarum numeratores sunt 1, 2, 3, 4, &c. transmutatis)

seriei B, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}$, &c. ∞ C+D+E+F, &c.

C. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ per praec. $\frac{1}{1}$

D. . $+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ C − $\frac{1}{2}$ ∞ $\frac{1}{2}$

E. . . $+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ D − $\frac{1}{6}$ ∞ $\frac{1}{3}$

F. . . . . . $+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ E − $\frac{1}{12}$ ∞ $\frac{1}{4}$

&c. ∞ &c.

∞ *G*; unde sequitur, seriem *G* ∞ *A*, totum parti, si summa finita esset.

Ego

Jakob Bernoulli was certainly convinced of the importance of his brother's deduction and emphasized its salient point when he wrote:

> The sum of an infinite series whose final term vanishes perhaps is finite, perhaps infinite.

Obviously, this proof features a naive treatment both of series manipulation and of the nature of "infinity." In addition, it attacks infinite series "holistically" as single entities, without recourse to the modern idea of partial sums. Before getting overly critical of its distinctly 17th-century flavor, however, we must acknowledge that Bernoulli devised this proof a century and a half before the appearance of a truly rigorous theory of series. Further, we can not deny the simplicity and cleverness of his reasoning nor the fact that, if bolstered by the necessary supports of modern analysis, it can serve as a suitable alternative to the standard proof.

Indeed, this argument provides us with an example of the history of mathematics at its best—paying homage to the past yet adding a note of freshness and ingenuity to the modern classroom. Perhaps, in contemplating this work, some of today's students might even come to share a bit of the enthusiasm and wonder that moved Jakob Bernoulli to close his *Tractatus* with the verse [7]

> So the soul of immensity dwells in minutia.
> And in narrowest limits no limits inhere.
> What joy to discern the minute in infinity!
> The vast to perceive in the small, what divinity!

Remark. Jakob Bernoulli, eager to examine other infinite series, soon turned his attention in section XVII of *Tractatus* to

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2}, \tag{4}$$

the evaluation of which "is more difficult than one would expect" ("difficilior est quam quis expectaverit"), an observation that turned out to be quite an understatement. He correctly established the convergence of (4) by comparing it termwise with the greater, yet convergent series

$$\begin{aligned} &1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots \\ &= 2\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) = 2(1) = 2. \end{aligned}$$

But evaluating the sum in (4) was too much for Jakob, who noted rather plaintively

> If anyone finds and communicates to us that which up to now has eluded our efforts, great will be our gratitude.

The evaluation of (4), of course, resisted the attempts of another generation of mathematicians until 1734, when the incomparable Leonhard Euler devised an enormously clever argument to show that it summed to $\pi^2/6$. This result, which Jakob Bernoulli unfortunately did not live to see, surely ranks among the most unexpected and peculiar in all of mathematics. For the original proof, see ([4], pp. 83–85). A modern outline of Euler's reasoning can be found in ([2], pp. 486–487).

REFERENCES

1. Jakob Bernoulli, *Ars Conjectandi*, Basel, 1713.
2. Carl B. Boyer, *A History of Mathematics*, Princeton University Press, 1985.
3. C. H. Edwards, *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.
4. Leonhard Euler, *Opera Omnia* (1), Vol. 14 (C. Boehm and G. Faber, editors), Leipzig, 1925.
5. Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972.
6. D. J. Struik (editor), *A Source Book in Mathematics (1200–1800)*, Harvard University Press, 1969.
7. Translated from the Latin by Helen M. Walker, as noted in David E. Smith's *A Source Book in Mathematics*, Dover, New York, 1959, p. 271.

> An idea reaches its full usefulness only when one understands it so well that one believes that one has always possessed it and becomes incapable of seeing it as anything but a trivial and immediate remark.
>
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