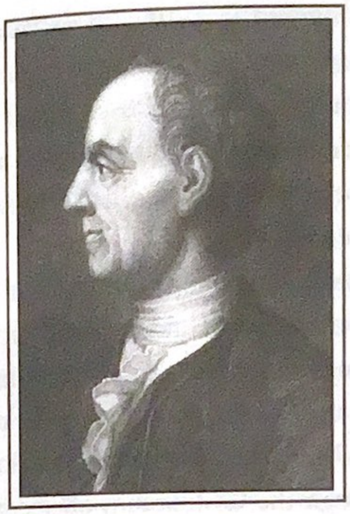

Leonhard Euler

(1707–1783)

HIS LIFE AND WORK

How long does it take to compose a life's work? That's an easy question to answer: a lifetime, of course! And how long does it take to publish a life's work? For Leonhard Euler, the answer is a century or more. The Swiss Academy of Science began to publish his collected works in 1911. To date, over six dozen volumes have been published. More remain. This should come as no surprise. A prominent historian of science has estimated that Euler produced nearly a quarter of all of the works in mathematics, physics, mechanics, astronomy, and navigation written in the eighteenth century. He was the most prolific mathematician of all time.

Like many of the mathematicians profiled in this book, Euler's genius cannot readily be traced back through his family tree. His mother, the former Margaretha Brucker, came from a long line of well-educated clergymen. His father, Paulus Euler, also a clergyman, had been the first member of the Euler family to be well-educated. Paulus's grandfather, Hans-Georg Euler, had been a combmaker when he settled in Basel in 1594. Paulus did have the distinction of having attended mathematics lectures given in Basel by the great Jacob Bernoulli (1654–1705), the first of eight great mathematicians from a family of hereditary genius.

Margaretha and Paulus married in 1701 and six years later, Leonhard Euler was born in the Swiss city of Basel on April 15. At that time, Basel was one of the thirteen Swiss republics. Its population of 17,000 consisted mostly of Calvinists, many of whom had immigrated from more orthodox Catholic environments across Europe. Among these immigrants were Jacob Bernoulli's forebears who had fled from Belgium in the late sixteenth century.

Shortly after his son's birth, Paulus took up a pulpit in the town of Riehen, five miles outside of Basel, where three more children were born to the family. Paulus served as his son's teacher, perhaps imparting some mathematical insights that had been gleaned at Jacob Bernoulli's lectures. At age eight, Leonhard went to Basel to live with his widowed maternal grandmother, Maria Magdalena Brucker-Farber, so that he could attend the Basel Gymnasium. Although the Gymnasium offered excellent instruction in Latin and Greek, it had no lectures in mathematics. The local citizenry thought mathematics too difficult for such young students. To fill this hole in the curriculum, Paulus hired the young theologian Johannes Burckhardt as a special tutor in mathematics for his son.

Five years later, Euler enrolled at the University of Basel at the age of thirteen. He pursued a theology degree while an undergraduate. The University's records from this period reveal that he showed unusual proficiency in the classics, being able to recite Virgil's 12,000 line *Aeneid* by heart. No record survives of any mathematical accomplishments from this time, although there must have been some.

When Leonhard received his Master's degree in 1723, he dutifully followed his father's wishes and continued on at the University to pursue a doctorate in theology. That was not to be. Euler soon became friends with Johann II Bernoulli (1710–1790) and together they attended the lectures of Johann I Bernoulli (1667–1748), the father of Johann II and brother of Jacob, whose lectures Paulus had attended. Under the influence of both Bernoullis, the young Euler immersed himself in mathematics. Johann I Bernoulli soon recognized Euler's immense talents and offered him private lectures. Johann I would soon write that Euler was

> *a young man of the most fortunate talents, from whose cleverness and acuteness we promise ourselves the greatest, having seen the ease and adroitness with which he has penetrated the most secret fields of higher mathematics under our auspices.*

Euler had many opportunities to exhibit the adroitness Bernoulli mentioned. In 1726, the Paris Academy sponsored a prize competition to determine the best way of setting up the masts on a sailing ship. Even though he had had no experience with sailing ships in landlocked Switzerland, his mathematical approach was so good that he came in second.

That same year, Euler completed his *Dissertation on the theory of sound* under the supervision of Bernoulli. This qualified him to apply for the vacant professorship of physics at the University. In spite of Bernoulli's ardent support for Euler, the University employed its standard method of drawing lots to reduce the field of applicants to just three. Euler's was not among the three lots drawn. Switzerland's loss would be Russia's gain.

In 1724, a year before his death, Tsar Peter the Great signed a decree founding a scientific academy in the new Russian capital St. Petersburg. Peter invited the most distinguished scholars across Europe to his new academy, among them were Niklaus II Bernoulli (1695–1726) and Daniel Bernoulli (1700–1782). As soon as they arrived in St. Petersburg, they recommended that the academy invite Euler, their father's most brilliant student. Although Euler would never return to Switzerland, he retained his Swiss citizenship for the rest of his life.

The Petersburg Academy offered Euler its chair in physiology. Euler quickly accepted and prepared for his new position by enrolling in Basel's medical faculty, three days before he embarked for Russia! In reality, Euler chose to ignore the details of his initial appointment. Instead, he lectured in mathematics, physics, and logic.

Euler would not be misplaced in the physiology chair for long. Daniel Bernoulli relinquished the physics chair to take up the mathematics chair when Jacob Hermann, its first holder, returned to Basel in 1730. That allowed Euler to take the vacant physics professorship. Two years later, Bernoulli himself returned to Basel, thereby allowing Euler to take the mathematics chair.

Peter the Great also had a practical end in mind when he founded the St. Petersburg Academy. He intended it to help modernize Russia. To that end, Euler made contributions in practical areas such as surveying, map-making, navigation, and, once again, naval construction.

Euler's income from these duties and his professorship allowed him to consider the prospect of marriage. He soon focused his attentions on Katharine Gsell, the daughter of a Swiss artist who also held an appointment at the Academy. Leonhard and Katharine married just after Christmas 1733. Their first son, Johann Albrecht, was born not quite a year later. The Eulers would have twelve more children born to them over the next seventeen years. Only five of them survived past early childhood.

Euler often commented that he made many of his greatest mathematical discoveries while holding a baby in one arm and with other children playing at his feet. Perhaps those were the circumstances when he hit upon the first of the discoveries presented in this chapter. It dates from the middle 1730s, shortly after he became a father for the first time.

The analysis of infinite series goes back to the ancient Greeks' consideration of Zeno's paradoxes, most notably in Book V of Aristotle's *Physics*. The new mathematics of the seventeenth century provided an arsenal of new tools to inaugurate a new age of analysis of infinite series.

In the late seventeenth century Jacob Bernoulli proved that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. He then raised the question of whether the sum of the inverse squares

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{m \to \infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2} \right)$$

diverges or converges. Bernoulli turned his attention to another infinite series in order to answer this question.

Triangular numbers are the numbers formed by the sum of the positive integers from 1 to n. For example, $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$. More generally

$$(n^2 + n)/2 = 1 + 2 + \cdots + (n - 1) + n$$

Bernoulli then proved that the series of inverse triangular numbers

$$\sum_{n=1}^{\infty} \frac{1}{\frac{n^2 + n}{2}}$$

converges to the value 2. Recognizing that $1/n^2 \leq 2/(n^2 + n)$ he easily proved that the sum of the inverse squares converges to a value no greater than 2. However, he was unable to determine that value. Bernoulli remarked, "If someone should succeed in finding what till now has withstood our efforts and communicate it to us, we shall be much obliged to him." This became known as the *Basel Problem* in honor of Bernoulli.

There is no record of Euler having studied number theory in Basel with his teacher Johann I Bernoulli, Jacob's younger brother. His interest in the subject seems to have been sparked in 1729 by Christian Goldbach, the first permanent secretary of the St. Petersburg Academy (and the person after whom Goldbach's conjecture, that every even number greater than 2 can be expressed as the sum of two primes, is named.)

Euler began to attack the Basel Problem in 1731. By 1735 he had solved it.

> *Now, however, against all expectation I have found an elegant expression for the sum of the seried 1 + 1/4 + 1/9 + 1/16 + etc, which depends on the quadrature of the circle ... I have found that six times the sum of this series is equal to the square of the circumference of a circle whose diameter is 1.*

The series summed to $\pi^2/6$! Here is an outline of his proof.

In 1715 the English mathematician Brook Taylor had demonstrated that the onometric sine function can be expressed as the infinite series.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Euler divided both sides of this equation by x resulting in

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

(Notice that $-1/3!$ is the coefficient of the term in x^2.) Since the sine function kes the value zero for all values $\pm n\pi$, the equation just given has non-zero roots $= \pm n\pi$ for $n = 1, 2, 3, \ldots$

Now Euler makes a daring leap. An n^{th} degree polynomial $P(x)$ having non-zero oots $a_1, a_2, a_3, \ldots, a_n$ and $P(0) = 1$ can be factored as follows.

$$P(x) = (1 - x/a_1) * (1 - x/a_2) * (1 - x/a_3) * (1 - x/a_n)$$

He leaps from this property of polynomials of finite degree to apply the prop- rty to the infinite degree polynomial just given with a sine function on the left hand de to boot! Having done that he arrives at

$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

Multiplying out the right side of this equation and collecting just the terms in x^2 yields

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

s the coefficient of x^2 in $\sin(x)/x$. Recalling that $-1/3!$ is the coefficient of the term in x^2, the left hand side can be replaced producing

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

or

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Quod erat demonstrandum!

Using the same methods, Euler also proved that the inverse fourth powers sum to $\pi^4/90$, the sum of the inverse sixth powers sum to $\pi^6/945$ and so on up to the sum of the inverse 26^{th} powers which equals $76977927 * (2^{24}/27!) * \pi^{26}$. Interestingly, none of the sums of the inverse odd powers is yet known.