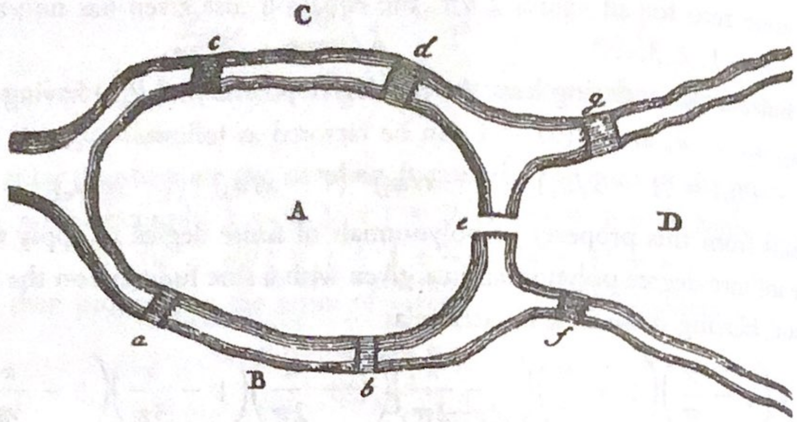
Euler fell ill with what he described as a "fiery fever" in 1735, shortly after finishing the work just described. He recovered to full health after a long recuperation. Neither the fever nor the long recuperation slowed Euler's mathematical output.

In 1736, Euler presented one of the first results in what he called the "geometry of position" (*geometria situs*), the study of relations dependent on position and not magnitude. Today, this is the branch of mathematics known as topology.

The two branches of the Pregel river met in the Prussian city of Königsberg, 600 miles southwest of St. Petersburg, at Kneihopf Island. Seven bridges crossed the river.

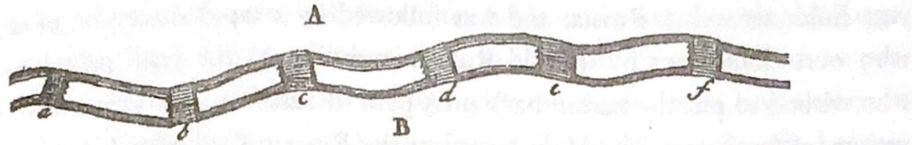


The people of Königsberg wondered if someone could take a walk crossing all of the bridges once, and only once. This became known as the *Seven Bridges of Königsberg Problem*. Most people doubted that such a walk was possible, but no one could prove it. Euler did! Our second Euler selection presents his analysis of this very problem.

Euler realized that he could solve the problem of the bridges simply by enumerating all possible walks that crossed bridges no more than once. However, that approach held no interest for him. Instead, he generalized the problem to consider

> *any configuration of the river and the branches into which it divides, as well as any number of bridges, to determine whether or not it is possible to consider each bridge exactly once.*

Euler commented that he would first consider the sequence of land areas traversed without considering which bridges were used. If **A**, **B**, **C**, and **D** are the land areas in the seven bridges problem, then **ABDC** represents a sequence of crossing from **A** to **B** to **D** to **C**. Euler begins his analysis by considering an extremely simple case: two land areas, **A** and **B**, connected by an arbitrary number of bridges.



If one bridge connects to land area **A**, then **A** can appear in a crossing sequence only once if that bridge is to be crossed only once. If two bridges connect to **A**, then **A** can appear in a crossing sequence only twice if each of those bridges is to be crossed only once. Generally, if an odd number N of bridges connect to **A**, then **A** can appear in a crossing sequence only $(N + 1)/2$ times if each of the bridges is to be crossed only once.

Returning to the particular details of the Königsberger's problem, Euler noted that an odd number of bridges connected to each of the land areas: 5 to **A**, and 3 to **B**, **C**, and **D** each. Given the result just described, the crossing must contain **A** three times and **B**, **C**, and **D** twice each, for a total of nine letters in the crossing sequence. However, if each of the seven bridges is to be crossed only once, such a crossing sequence can only contain **A**, **B**, **C**, and **D** a total of eight times. Although Euler had solved the Königsberger's problem, he did not stop there.

He continued by noting that if an even number N of bridges connect to **A**, then **A** can appear in a crossing sequence only $\frac{1}{2}N + 1$ times if **A** is a starting point for the journey, and only $\frac{1}{2}N$ times if **A** is not a starting point for the journey (if each of the bridges is to be crossed only once.) Having established that Euler recognizes that

> *Every route must, of course, start in some region, thus from the number of bridges that lead to each region I determine the number of times the corresponding letter will occur in the expression for the entire route as follows: When the number of bridges is odd I increase it by one and divide by two; when the number if even I simply divide it by two. Then if the sum of the resulting numbers is equal to the actual number of bridges plus one, the journey can be accomplished, though it must start in a region approached by an odd number of bridges. But if the sum is one less than the number of bridges plus one, the journey is feasible if starting from a region approached by an even number of bridges, for in that case the sum is again increased by one.*

Modern topology now treats land areas and bridges as vertices and edges connecting vertices.

Euler's fever returned shortly after he finished his work on the Seven Bridges problem. By 1738, he had lost vision in his right eye.

Meanwhile, Peter the Great's widow, the Empress Catherine I, was continuing her husband's liberalizing from the Russian throne. However, she died in 1727, the year Euler arrived in Russia, and was followed by a rapid succession of weak rulers who were dominated by the old Russian nobility. As the years passed, the nobles, who wished to put the nation back on a path of russification, became ever more suspicious of foreigners. The Holy Synod of the Russian Orthodox Church weighed in against the academy because of its teaching of the new sciences in general and Copernican astronomy in particular.

As Russia was pursuing a path of de-liberalization, a new king to the west ascended his throne planning on liberalizing his country. Frederick II, also known as the Great, ascended the Prussian throne in 1740 and immediately made plans to resurrect the scientific academy in his capital Berlin to be the equal of the academies in Paris, London, and St. Petersburg. Euler received one of the first invitations to join the new academy.

Euler quickly accepted the invitation. The new Director of the St. Petersburg Academy, Johann Schumacher, was a bureaucrat whose main goal seemed to be "the suppression of talent wherever it might rear its inconvenient head," according to Euler. In his words, Russia had become "a country where any person who speaks is hanged." Euler's wife was also eager to leave St. Petersburg for fear of the frequent fires that might consume their wooden house.

Euler left St. Petersburg in June 1741 having published over fifty books and articles during his fourteen years in the new Russian capital. He arrived in Berlin a month later and would remain in Berlin for a quarter century.

While in Berlin, he continued his outpouring of work. He was so prolific that his new papers, about 380 of them, filled the pages of the journals of the academies in both Berlin (in French) and St. Petersburg (in Latin). During this period, Euler also wrote a number of popular textbooks. His *Mechanica* was the first work to present Newton's laws of motion using the framework of the new infinitesimal calculus.

Euler's *Introduction to Analysis of the Infinite,* first published in 1748, is his best known textbook from this period. In this two-volume work, Euler presented the sum of the inverse squares as well as virtually every other result in mathematical analysis then known without using the techniques of the new and still controversial infinitesimal calculus. This text introduced generations of readers to the symbols cos and sin for the trigonometric functions, π to represent the ratio of a circle's circumference to its diameter, i for $\sqrt{-1}$, and finally the eponymous e for the base of the natural logarithms. It is no wonder that many historians of mathematics consider the *Introduction* to be the equal of Euclid's *Elements* in terms of its impact.

Euler's best selling book from this period arose from his providing instruction in elementary science to the young Princess of Anhalt-Dessau. Euler published these lessons as *Letters of Euler on Different Subjects in Natural Philosophy Addressed to a*

German Princess. It contains over 200 letters on fields as diverse as logic, language, astronomy, gravity, light, sound, and magnetism. In addition, there are letters on particular questions such as why the sky is blue, why the moon looks larger when it rises, and why it is cold at the top of a mountain in the tropics. Finally, because Euler was also responsible for providing moral training to the Princess, the *Letters* include his thoughts on the origins of evil and the conversion of sinners.

In addition to his writing, Euler also found time to perform many other, more practical works. The King assigned Euler many tasks for the kingdom's benefit. These included correcting the level of the Finow canal, a major conduit for commerce in eastern Prussia, organizing state lotteries, and advising the government on issues concerning insurance, annuities, and widow's pensions. Euler also supervised the academy's observatory and its botanical gardens. Frederick even asked Euler to supervise work on the plumbing system at Sans Souci, the royal summer residence.

Euler hoped to be named the Academy's president after its first president, Pierre de Maupertuis, died in 1759. After all, he was the greatest scientist in Europe at mid-century. However, Euler lacked several key qualifications required by the king. He was Swiss rather than French. Moreover, by then Euler and his family were living on a farm outside of Berlin, hardly a sign of the worldly polish that Frederick greatly valued. In the end King Frederick named another Frenchman, Jean le Rond d'Alembert, to be the academy's new president. The disappointed Euler immediately determined to leave Berlin.

Euler had remained on extremely cordial terms with his former colleagues in St. Petersburg throughout his quarter century in Berlin. In recognition of the services he had previously rendered, the Academy in St. Petersburg sent him a regular stipend, even while Russia and Prussia were engaged in the Seven Years War.

By the mid-1760s the ascent of Empress Catherine the Great had restored enough political order to Russia that Euler gladly accepted an offer to return to St. Petersburg of course, Euler needed Frederick's assent to leave Berlin. When he first wrote the King asking for his release, the King simply refused to answer. After two more letters of resignation, the King wrote back that Euler should withdraw his resignation and stop bothering him. Only Catherine the Great's intervention persuaded Frederick to release Euler. The King did this by sending a terse note that read "With reference to your letter of April 30, I am giving you permission to resign in order to go to Russia."

The Eulers returned to St. Petersburg in the summer of 1766 and moved into a house on the embankment of the River Neva, not far from the Academy. Back in St. Petersburg the Eulers' life would not be easy. Leonhard had developed a cataract in his good left eye during his last years in Berlin. It became much more severe after he returned to Russia and decided to have the occluded lens removed surgically. At first it appeared as though the surgery had been a success. However, complications arose and Euler soon lost the use of his left eye, leaving him with only the most meager vision for the rest of his life.

Shortly afterwards Katherine's worst fears would be realized. In May 1771 a terrible fire swept through St. Petersburg destroying over 500 houses along the Neva including the Eulers'. Only the efforts of Peter Grimm, a brave craftsman from Basel, saved the nearly blind Euler and his manuscripts from perishing as his house went up in flames.

Two and a half years later Euler became a widow when in November 1773 his wife suddenly died at the age of 66. Euler persevered in his work in spite of the loss of his sight and his wife. In 1775 he wrote an average of one mathematics paper per week, dictating them to a scribe. No doubt Euler's prodigious memory, which had enabled him to memorize the *Aeneid* in his teens, made it possible for him to continue his research. The second example of Euler's work presented here, his proof that every integer can be expressed as the sum of no more than four squares, dates from this period.

In 1621 the French mathematician Claude Bachet conjectured that every positive integer can be expressed as the sum of no more than 4 squares. The great mathematical amateur Pierre de Fermat made some progress on this problem but was unable to provide a satisfactory proof. Euler did.

In his proof Euler makes use of modular arithmetic. His proof depends critically on the fact that if k is not divisible by N, then $(aN + k)^2$ and $(aN - k)^2$ have the same remainder when divided by N. Therefore, if N is an odd prime then half of the numbers of the form $aN + b$ for $b = 1, 2, \ldots, N - 1$ are squares modulo N and half are not.

At about the same time that Euler published this proof the English mathematician Edward Waring extended Bachet's original conjecture and asserted without proof that every positive integer can also be expressed as the sum of no more than 9 cubes, no more than 19 fourth powers, and that in general, for any power K there is a value S such that every positive integer can be expressed as the sum of no more than S K^{th} powers. In 1909 the German mathematician David Hilbert proved Waring's conjecture!

Euler lived another decade following his wife's death, marrying her half-sister in July 1776. During early September 1783 he began to suffer from bouts of dizziness. The 7th of that month began as a normal day for Euler, spending the morning tutoring one of his grandchildren and then working out calculations for the orbit of the recently discovered planet Uranus. Later in the day while taking tea and smoking a pipe, the pipe fell out of his hands. When he tried to pick up his pipe he failed and soon collapsed crying "I am dying." He had suffered a stroke. He died later that night without regaining consciousness. It took the Academy's journal over 20 years to publish all of his remaining manuscripts.