Chapter Eight

Ancient Indian Mathematics

A Restatement of Intent and a Brief Historical Sketch

Ancient Indian history raises many problems. The period before the Christian era takes on a haziness that seems to have prompted opposing reactions. There are those who make excessive claims for the antiquity of Indian mathematics, and others who go to the opposite extreme and deny the existence of any "real" Indian mathematics before about AD 500. The principal motive of the former is to emphasize the uniqueness of Indian mathematical achievements. In this view, if there was any influence, it was always a one-way traffic from India to the rest of the world. The motives of the latter are more mixed. For some their Eurocentrism (or Graeco-centrism) is so deeply entrenched that they cannot bring themselves to face the idea of independent developments in early Indian mathematics, even as a remote possibility.[1]

A good illustration of this blinkered vision is provided by a widely respected historian of mathematics at the turn of the twentieth century, Paul Tannery. Confronted with the evidence from Islamic sources that the Indians were the first to use the sine function as we know it today, Tannery devoted himself to seeking ways in which the Indians could have acquired the concept from the Greeks. For Tannery, the very fact that the Indians knew and used sines in their astronomical calculations was sufficient evidence that they must have had it from the Greeks.[2] But why this tunnel vision? The following quotation from G. R. Kaye (1915) is illuminating:

> The achievements of the Greeks in mathematics and art form the most wonderful chapters in the history of civilisation, and these achievements are the admiration of western scholars. It is therefore natural that western investigators in the history of knowledge should seek for traces of Greek influence in later manifestations of art, and mathematics in particular.

It is particularly unfortunate that Kaye is still quoted as an authority on Indian mathematics. Not only did he devote much attention to showing the derivative nature of Indian mathematics, usually on dubious linguistic grounds (his knowledge of Sanskrit was such that he depended largely on indigenous *pandits* for translations of primary sources), but he was prepared to neglect the weight of contemporary evidence and scholarship to promote his own viewpoint. So, while everyone else claimed that the Bakhshali Manuscript (discussed at the end of this chapter) was written or copied from an earlier text dating back to the first few centuries of the Christian era, Kaye insisted that it was no older than the twelfth century AD. Again, while the Islamic sources unanimously attributed the origin of our present-day numerals to the Indians, Kaye was of a different opinion. And the distortions that resulted from Kaye's work have to be taken seriously because of his influence on Western historians of mathematics, many of whom remained immune to findings that refuted Kaye's inferences and established the strength of the alternative position much more effectively than is generally recognized.

This tunnel vision is not confined to mathematics alone. Surprised at the accuracy of information on the preparation of alkalis contained in an early Indian textbook on medicine (*Sushruta Samhita*)[3] dating back to a few centuries BC, an eminent chemist and historian of the subject, M. Berthelot (1827–1909), suggested that this was a later insertion, after the Indians had come into contact with European chemistry!

While non-European chauvinism (on the part of, for example, the Arabs, Chinese, and Indians) does persist, "arrogant ignorance"—as J. D. Bernal (1969) described the character of Eurocentric scholarship in the history of science—is the other side of the same coin. But the latter tendency has done more harm than the former because it rode upon the political domination imposed by the West, which imprinted its own version of knowledge on the rest of the world.

Table 8.1 offers a brief summary of the main events in the long history of India as a backdrop to the development of mathematics; it divides Indian history up to the beginning of the sixteenth century into six periods. The map of India in figure 8.1 shows places mentioned in the text. The earliest evidence of mathematics is found among the ruins of the Indus Valley civilization, which goes back to 3000 BC. (It is perhaps more appropriately referred to as the Harappan civilization, since at its peak it spread far beyond the Indus Valley itself.) Around 1500 BC, according to the traditional —though increasingly contentious—view among historians, a group of

TABLE 8.1: CHRONOLOGY OF INDIAN HISTORY AND MATHEMATICS

Period	*Main historical events*	*Mathematics*	*Notable mathematicians*
3000–1500 BC	The Indus Valley civilization (script undeciphered) covering 1–2 million square km; main urban centers Harappa, Lothal, and Mohenjo-Daro	Weights, artistic designs, "Indus scale"; brick technology probably influenced the construction of Vedic altars in the next period	
1500–500 BC	The coming of the Aryans; the formation of Hindu civilization; the emergence of the *Code of Manu*; the recording of the Vedas and *Upanishads*	*Vedangas* and *Sulbasutras*; problems in astronomy, arithmetical operations, Vedic geometry	Baudhayana, Apastamba, Katyayana
500–200	The establishment of Indian states; the rise of Buddhism and Jainism; contacts with Persia maintained; the Mauryan empire, culminating in the reign of Asoka, who spread Buddhism abroad	Vedic mathematics continues during the earlier years but declines with ending of ritual sacrifices; beginnings of Jaina mathematics: number theory, permutations and combinations, the binomial theorem; astronomy	
200 BC–AD 400	Triple division: Kushan dynasty (North), Pandyas (South), Bactrian-Persian (Punjab); pervading influence of Buddhism in art and sculpture	Jaina mathematics: rules of mathematical operations, decimal-place notation, first use of 0; algebra including simple, simultaneous, and quadratic equations; square roots; details of how to represent unknown quantities and negative signs	

continued

Table 8.1: Continued

Period	*Main historical events*	*Mathematics*	*Notable mathematicians*
400–1200	Imperial Guptas reaching their height in the reign of Harsha (606–647); flowering of Indian civilization as shown in science, philosophy, medicine, logic, grammar, and literature	The Classical period of Indian mathematics; important works: the Bakshali Manuscript, *Aryabhatiya*, *Panca-siddhantika*, *Aryabhatiya Bhasya*, *Maha Bhaskariya*, *Brahma Shputa-siddhanta*, *Patiganita*, *Ganita Sara Samgraha*, *Ganitilaka*, *Lilavati*, *Bijaganita*	Aryabhata I, Varahamihira, Bhaskara I, Brahmagupta, Sridhara, Mahavira, Bhaskara II (also known as Bhaskaracharya)
1200–1600	Early Muslim dynasties; birth of Sikhism; the Hindu kingdom of Vijaynagar in the South	Decline of mathematics and learning in the North; the rise of the Kerala school of astronomy and mathematics; work on infinite series and analysis	Narayana, Madhava, Nilakantha

people descended from the north and destroyed the Harappan culture, but not before they had absorbed some of its features. These invaders are often referred to as “Aryans”—a term that has acquired an unfortunate connotation in modern times through its association with the Nazis.

The Aryans were a pastoral people, speaking a language that belonged to the Indo-European family. It remained for a long time a spoken rather than a written language, with writing initially restricted to the vernaculars. Over the years this language, Sanskrit, developed sufficiently to become a suitable medium for religious, scientific, and philosophical discourse. Its potential for scientific use was greatly enhanced as a result of the thorough systematization of its grammar by Panini, about 2,600 years ago. In a book titled *Astadhyayi* (Eight Chapters), Panini offered what must be the first attempt at a structural analysis of a language. On the basis of just under four thousand sutras (i.e., rules expressed as aphorisms), he built virtually

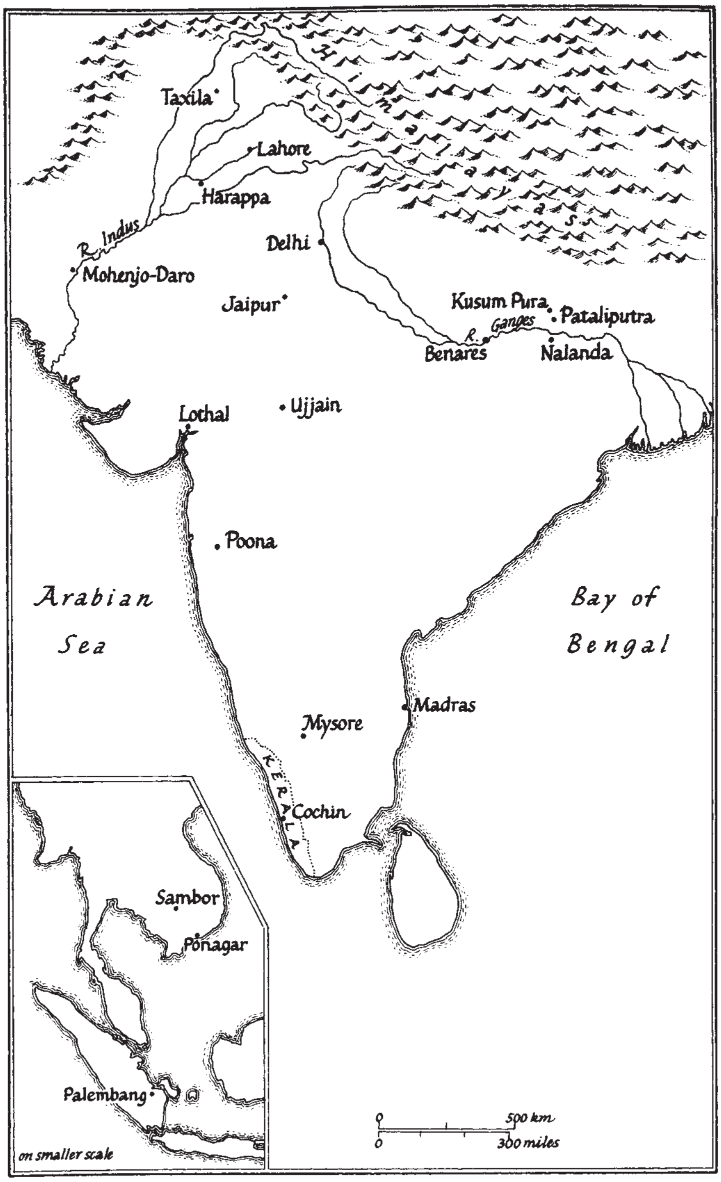


FIGURE 8.1: Map of India and (inset) Southeast Asia

the whole structure of "Classical" Sanskrit language, whose general "shape" hardly changed for the next two thousand years. Sanskrit served as a useful medium for recording early scriptural texts such as the Vedas and *Upanishads*, early scientific literature such as the *Vedangas* (or Limbs of the Vedas), and early rules of social conduct such as the *Code of Manu*.

An indirect consequence of Panini's efforts to increase the linguistic facility of Sanskrit soon became apparent in the character of scientific and mathematical literature. This may be brought out by comparing the grammar of Sanskrit with the geometry of Euclid—a particularly apposite comparison since, whereas mathematics grew out of philosophy in ancient Greece, it was, as we shall see, partly an outcome of linguistic developments in India.

The geometry of Euclid's *Elements* starts with a few definitions, axioms, and postulates and then proceeds to build up an imposing structure of closely interlinked theorems, each of which is in itself logically coherent and complete. In a similar fashion, Panini began his study of Sanskrit by taking about seventeen hundred basic building blocks—some general concepts, vowels and consonants, nouns, pronouns and verbs, and so on—and proceeded to group them into various classes. With these roots and some appropriate suffixes and prefixes, he constructed compound words by a process not dissimilar to the way in which one specifies a function in modern mathematics. Consequently, the linguistic facility of the language came to be reflected in the character of mathematical literature and reasoning in India. Indeed, it may even be argued that the algebraic character of ancient Indian mathematics is but a by-product of the well-established linguistic tradition of representing numbers by words.

The third period of Indian history began around 800 BC. It saw not only the establishment of two of the great religions originating in India, Buddhism and Jainism, but also the growth of independent states, a number of which were later merged to form the first of the great empires of India, the Mauryan empire. This period marked the decline of Vedic mathematics and the gradual emergence of the Jaina school, which was to do notable work in number theory, permutations and combinations, as well as other abstract areas of mathematics.

The fourth period, from about 200 BC, was a period of instability and fragmentation brought about by waves of foreign invasions. But it was also a time of useful cross-cultural contacts with neighbors and with the Hellenistic world, bringing fresh ideas into Indian science and laying the foundation for great advances in the next period. The Kushan empire became an important vehicle for spreading not only Buddhist religion and art but also Indian science, particularly astronomy, into western Asia. Probably the only piece of existing mathematical evidence from this period is the Bakhshali Manuscript. However, the earlier dating of this manuscript to the third century is based on an estimate made by Hoernle, who was the first to study it. On the basis of recent evidence, notably that of Hayashi (1995), the manuscript cannot be dated earlier than the eighth century.

The fifth period, from the third to the twelfth centuries, is often referred to as the Classical period of Indian civilization. The earlier part of this period saw much of India ruled by the imperial Guptas, who encouraged the study of science, philosophy, medicine, and other arts. Mathematical activities reached a climax with the appearance of the famous quartet: Aryabhata, Brabmagupta, Mahavira, and Bhaskaracharya. Their lives and works will be examined in the next chapter. Indian work on astronomy and mathematics spread westward, reaching the Islamic world, where it was absorbed, refined, and augmented before being transmitted to Europe.

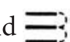
The last period, which we may describe as the "medieval" period of Indian history, saw the rise of great states in southern India and a migration of mathematics and astronomy from the North to the South, probably as a result of political upheavals. It was believed for a long time that mathematical development came virtually to a stop in India after Bhaskaracharya in the twelfth century. There may be some element of truth in this as far as the North was concerned, but in the South—and particularly in the Southwest, in the area corresponding to the present-day state of Kerala—this was a period marked by remarkable studies of infinite series and mathematical analysis that predated similar work in Europe by about three hundred years.

The mathematics of Kerala will be presented in a separate chapter. In this chapter we examine Indian mathematics from its early beginnings to just before the Classical period; in the next chapter we consider mainly Classical Indian mathematics. The development of Indian numerals is dealt with in this chapter, though there is some historical overlap, particularly when one considers the spread of the numerals into countries such as Cambodia and Java to the east, and into the Islamic world to the west. The reader may wish to refer to table 8.1 and figure 8.1 whenever necessary to sketch in the historical and geographical background to this and the next chapter.

Math from Bricks: Evidence from the Harappan Culture

Between 1921 and 1923 a series of archaeological excavations along the banks of the Indus uncovered the remains of two urban centers, at Harappa and Mohenjo-Daro, dating back to about 3000 BC. Subsequent searches over the last four decades have revealed further remains spread across an

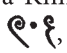
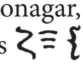
Early Indian Numerals and Their Development

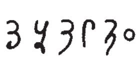
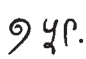
Three early types of Indian numerals are shown in table 8.2 in chronological order of appearance. The Kharosthi-type numerals, derived from the Aramaic script, are found in inscriptions dating to a period from the fourth century BC to the second century AD. Special symbols were used to show both 10 and 20. Numbers up to 100 were then built up additively; for larger numbers the multiplication principle came into operation, with special symbols for higher powers of 10. Following from their West Asian origins, the Kharosthi numerals were written from right to left. The most complete example of this type of numerals is the Saka numerals from around the first century BC. The Brahmi-type numerals were more highly developed. There were separate symbols for the digits 1, 4 to 9, and the number 10 and its higher powers. There were also symbols for multiples of 10 up to 90, and for multiples of 100 up to 900. The number 486, for example, would be written by using the symbols for 400, 80, and 6. It is possible that our symbols "2" and "3" are cursive versions of the Brahmi numerals (i.e., from  and  may have evolved 2 and 3).

The earliest trace of Brahmi-type numerals is from the third century BC, on the Asoka pillars scattered around India, though more detailed pieces of evidence are found elsewhere later. At the top of Nana Ghat near Poona in central India is a cave that must once have been a resting place for travelers; inscribed on the cave walls are numerals representing the signs for 10 and 7, which date back to 150 BC. Another version of the Brahmi numerals (shown in table 8.2) is found at Nasik, near present-day Bombay (now Mumbai), from around 100 BC. Both versions resemble each

Table 8.2: Three Types of Indian Numerals, in Chronological Order

Note: The Kharosthi numeral for 9 is not known for certain other, and it was thought until recently that from them evolved first the Bakhshali number system (c. AD 400–1200) and then the Gwalior system (c. AD 850), which is recognizably close to our present-day number system.[13] In both the Bakhshali and Gwalior number systems, ten symbols were used to represent 1 to 9 and zero. With them it became possible to express any number, no matter how large, by a decimal place-value system.

The earliest appearance of the symbol that we associate with zero in India in a decimal place-value system is in an inscription from Gwalior dated “Samvat 933” (AD 876), where the numbers 50 and 270 are given as  and  respectively. Note the close similarity with our notation for 270. For earlier evidence, we have to turn to Southeast Asia when it was under the cultural influence of India. There, three inscriptions have been found bearing dates in the Saka era, which began in AD 78. A Malay inscription at Palembang in Sumatra from AD 684 shows 60 and 606 Saka as  and  respectively, a Khmer inscription at Sambor in Cambodia from AD 683 gives 605 as , and an inscription at Ponagar, Champa (now southern Vietnam), from AD 813 represents 735 as . If, however, the original version of the Bakhshali Manuscript dates from the third century AD, it would be the earliest evidence of a well-established number system with a place-value scale and zero that is also recognizably an ancestor of our present-day number system. In the Bakhshali Manuscript are found the following numbers:

330: , 846,720: , 947: .

What we have here is a fully developed decimal place-value system incorporating zero.

The Emergence of the Place-Value Principle

Fascination with numbers has been an abiding characteristic of Indian civilization. Not only large numbers but very small ones as well. Operations with zero attracted the interest of both Bhaskaracharya (b. 1114) and Srinivas Ramanujan (1887–1920). In an elementary class that Ramanujan attended, the teacher was explaining the concept of division (or “sharing”) through examples: between three children, each child would get one banana. Similarly, the share would be one banana if four bananas were shared among four children, five bananas among five children, and so on. And when the teacher generalized this idea of sharing *x* bananas among *x* boys, Ramanujan asked whether, if *x* equaled zero, each child would then get one banana! There is no record of the teacher's reply. Ramanujan explained later to his school friends that zero divided by zero could be anything, since the zero of the denominator may be any number of times the zero of the numerator.

Two important features of early numeration may have been of significance in the subsequent development of Indian numerals. Ever since the Harappan period the number 10 may have formed the basis of numeration; there is no evidence of the use of any other base in the whole of Sanskrit literature. Long lists of number-names for powers of 10 are found in various early sources. For example, one of the four major Vedas, the *Yajur-veda*, gives special names for powers of ten from one or 10^0 (*eka*) to one trillion or 10^{12} (*parardha*). In the *Ramayana*, one of the most popular texts of Hinduism and roughly contemporaneous with the later Vedas, it is reported that Ravana, the chief villain of the piece, commanded an army whose total equaled $10^{12} + 10^5 + 36(10^4)$. Facing them was the rival army of Rama, the hero of the epic, which had $10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} + 10^{40} + 10^{44} + 10^{52} + 10^{57} + 10^{62} + 5$ men! Even though these numbers are fantastic, the very existence of names for powers of ten up to 62 indicates that the Vedic Indians were quite at home with very large numbers. This is to be compared with the ancient Greeks, who had no words for numbers above the myriad (10^4).

And these were by no means the largest numbers ever conceived in ancient India. The Jains, who came after the Vedic Indians, were particularly fascinated by even larger numbers, which were intimately tied up with their philosophy of time and space. This fascination with large numbers is also found in Buddhist literature. In the life of the Buddha, as reported in *Lalita-vistara,* the young Buddha, as part of a competion to win the hand of the princess Gopa, recites a table that includes names for powers of 10 going up to the fiftieth power.[14] (We shall look at the Jaina contribution in detail in a later section.) For units of measuring time, the Jains suggested the following relationships:

$$1\ \textit{purvis} = 756 \times 10^{11} \text{ days;}$$
$$1\ \textit{shirsa prahelika} = (8{,}400{,}000)^{28}\ \textit{purvis}.$$

The last number contains 194 digits!

The early use of such large numbers eventually led to the adoption of a series of names for successive powers of 10. The importance of these number-names in the evolution of the decimal place-value notation cannot be exaggerated. The word-numeral system, later replaced by an alphabetic notation, was the logical outcome of proceeding by multiples of 10. Thus 60,799 is *sasti* (sixty) *sahasra* (thousand) *sapta* (seven) *sata* (hundred) *navati* (nine ten times) *nava* (nine). Such a system presupposes a scientifically based vocabulary of number-names in which the principles of addition, subtraction, and multiplication are used. It requires:

1. The naming of the first nine digits (*eka*, *dvi*, *tri*, *catur*, *pancha*, *sat*, *sapta*, *asta*, *nava*)

2. A second group of nine numbers obtained by multiplying each of the first nine digits by ten (*dasa*, *vimsati*, *trimsat*, *catvarimsat*, *panchasat*, *sasti*, *saptati*, *asiti*, *navati*)

3. A group of numbers that are increasing integral powers of 10, starting with 10^2 (*sata*, *sahasra*, *ayuta*, *niyuta*, *prayuta*, *arbuda*, *nyarbuda*, *samudra*, *madhya*, *anta*, *parardha* . . .).

In forming the words of the second and third groups of numbers, the multiplicative principle applies, as in the example quoted: 60,000 is *sasti-sahasra.* The additive principle is employed when the numbers from the first and second group are used, for example, 27 is *sapta-vimsati.* The subtractive principle may apply occasionally and in a limited way; for example, *ekanna-catvarimsat* indicates $40 - 1 = 39$, where *ekanna* means "one less."

To understand why word-numerals persisted in India, even after the Indian numerals became widespread, it is necessary to recognize the importance of the oral mode of preserving and disseminating knowledge. An important characteristic of written texts in India from time immemorial was the sutra style of writing, which presented information in a cryptic form, leaving out details and rationale to be filled in by teachers and commentators. In short pithy sentences, often expressed in verses, the sutras enabled the reader to memorize the content easily.

As a replacement for the older word-numeral system that consisted of merely names of numbers, a new system (a concrete number system) was devised to help versification and memory. In this system, known as *bhuta-samkhya*, numbers were indicated by well-known objects or ideas. Thus, zero was *shunya* (void) or *ambara akasa* (heavenly space or sky or ether) or other empty things, one was *candra* (moon) or *bhumi* (earth) or other single things, two was *netra* (eyes) or *paksa* (wings of a bird) or other pairs, three was *kala* (time: past, present, and future) or *loka* (heaven, earth, and hell) or other trios, and so on. With multiple words available for each number, the choice of a particular word for a number would be dictated by literary considerations. This form of notation continued for many years in both secular and religious writings because it was aesthetically pleasing and offered an easier way of remembering numbers and rules.

There were two major problems with the *bhuta-samkhya* system. First, there was an "exclusionist" element, in that to decode the words for their numerical values required considerable familiarity with the philosophical and religious texts from which the correspondences were established in the first place. Second, at times the same word stood for two or more different numbers, since some writers had their own preferences when it came to choosing words to correspond to numbers as, for example, when *paksa* was used for 2 as well as 15 and *dik* for 8, 10, and 4.

There are traces of this system of numeration in the *Yavanajataka* (AD 269) of Sphujidhvaja, although the first clearest and detailed evidence of it is found in the works of the astronomer Varahamihira (d. AD 587). Thus, except for the actual symbols themselves, the present-day number system with distinct numerals for the numbers from zero to 9, the place-value principle, and the use of the zero within the decimal base is essentially what we see in this early number system.[15] In a sense, what is used as a symbol for a number, whether it be a letter, a word, or a specially invented squiggle, is of little importance. Indeed, an unduly close association—or even identity—between a number and the symbol used to represent it may even be counterproductive, preventing the strength of the place-value principle from being fully exploited in elementary operations.

A third system of numerical notation originated with Aryabhata (b. 476 AD). In his *Aryabhatiya,* he introduced an alphabetical scheme for representing numerals, based on distinguishing between classified (*varga*) and unclassified (*avarga*) consonants and vowels. The *vargas* fall into five phonetic groups: *ka-varga* (guttural), *ca-varga* (palatal), *ta-varga* (lingual), *ta-varga* (dental), and *pa-varga* (labial). Each group has five letters associated to it, and represented numbers from 1 to 25. There were seven *avargas* consisting of semivowels and sibilants representing numerical values 30, 40, 50, . . . , 190. An eighth *avarga* was used to extend the number to the next place value. The ten vowels denoted successive integral powers of 10 from 100 onward.

This form of representation, closer to the system that preceded *bhuta-samkhya,* has the advantage of brevity and clarity but the disadvantage of having limited potential for formations of words that are pronounceable and meaningful, both necessary requirements for easy memorization. For example, in the Aryabhatan system, the representation of the number of revolutions of the moon in a *yuga* (calculated as 57,753,336 days) is the unpronounceable and meaningless word *cayagiyinusuchlr*!

From a refinement of Aryabhata's alphabet-numeral system of notation emerged the *katapayadi* system, which the legendary founder of the Kerala school of astronomy, Varurici, was believed to have popularized around the fourth century AD. In this system, every number in the decimal place-value system can be represented by words, each letter of the word representing a digit. A vowel not preceded by a consonant stands for zero, but vowels following consonants have no special value. In the case of conjunct consonants (a combination of two or more consonants), only the last consonant has a numerical value. Number-words are read from right to left so that the letter denoting the "units" is given first, and so on.

This was a system devised to help memorization, since memorable words can be made up using different chronograms. For example, if such a system is applied to English, the letters *b*, *c*, *d*, *f*, *g*, *h*, *j*, *k*, *l*, *m* would represent the numbers zero to 9. So would *n*, *p*, *q*, *r*, *s*, *t*, *v*, *w*, *x*, *y*. The last letter, *z*, denotes zero. The vowels, *a*, *e*, *i*, *o*, *u* are helpful in forming meaningful words but have no numerical values associated with them. Thus, the sentence "I love Madras" represents the numbers 86 and 9,234. To take another example from Kunjunni Raja (1963, p. 123), the number 1,729,133 could be represented by *balakalatram saukhyam* (i.e., the [company] of a young woman is sheer happiness) or *lingavyadhir asahyah* (i.e., the demise of sexual virility is unbearable).

The close relationship between literacy and numeracy, implied by such varied systems of numerical notation, may have its roots in the way that Sanskrit developed in its formative period after its separation from other languages of the Indo-European family. A long tradition of oral communication of knowledge was a characteristic of that period and left a singular mark on the nature and transmission of knowledge, whether religious or scientific, in Indian culture. After many years, as Sanskrit became a written language, three kinds of scientific Sanskrit developed with varying degrees of artificiality: grammatical, logical, and mathematical Sanskrit.

Mathematical Sanskrit remained the least artificial of the three, with the greatest artificiality found in the development of grammatical Sanskrit by Panini and Patanjali, followed five hundred years later by the logical

Sanskrit of *Nyaya*, which culminated a thousand years later in *Navya-Nyaya*. This has important implications for a comparative study of the historical development of Indian and Western mathematics, according to Staal (1995). First, the chronological order of the development of artificial scientific languages in the West was a reversal of the Indian experience. In the West, logic followed mathematics, and linguistics was a late developer. In India, mathematical Sanskrit never quite became an artificial language, although it employed abbreviations and artificial notations outside Sanskrit as shorthand for practical procedures. And logical Sanskrit never became, like its Western counterpart, an important adjunct to "mathematical philosophy."

The Enormity of Zero

The word "zero" comes from the Arabic *al-sifr*.[16] *Sifr* in turn is a transliteration of the Sanskrit word *shunya*, meaning void or empty, which later became the term for zero. Introduced into Europe during the Italian Renaissance in the twelfth century by Leonardo Fibonacci (and by Nemorarius, a less well-known mathematician) as *cifra*, the word emerged in English as "cipher." In French it became *chiffre*, and in German *ziffer*, both of which mean zero.

The ancient Egyptians never used a zero symbol in writing their numerals. Instead they had a stand-alone zero to represent a benchmark value or magnitude. A bookkeeper's record from the Thirteenth dynasty (about 1700 BC) shows a monthly balance sheet for items received and disbursed by the royal court during its travels. On subtracting total disbursements from total income, a zero remainder was left in several columns. This zero remainder was represented by the hieroglyph *nfr*, which also means beautiful or complete in ancient Egyptian. The same *nfr* symbol also labeled a zero reference point for a system of integers used on construction guidelines at Egyptian tombs and pyramids. These massive stone structures required deep foundations and careful leveling of the courses of stone. A vertical number-line labeled the horizontal leveling lines that guided construction at different levels. One of these horizontal lines, often at pavement level, was used as a reference and was labeled *nfr* or zero. Horizontal leveling lines were spaced 1 cubit apart. Those above the zero level were labeled as 1 cubit above *nfr*, 2 cubits above *nfr*, and so on. Those below the zero level were labeled 1 cubit below *nfr*, 2 cubits below, and so forth. Here zero was used as a reference for directed or signed numbers.

It is quite extraordinary that the Mesopotamian culture, more or less contemporaneous to the Egyptian culture, developed a full positional-value number system on base 60 and did not use zero as a number. A symbol for zero as a placeholder appeared late in the Mesopotamian culture. The early Greeks, who were the intellectual inheritors of Egyptian mathematics and science, emphasized geometry to the exclusion of everything else. They did not seem interested in perfecting their number notation system. They simply had no use for zero. In any case, they were not greatly interested in arithmetic, claiming that arithmetic should only be taught in democracies, for it "dealt with relations of equality." On the other hand, geometry was the natural study for oligarchies, for "it demonstrated the proportions within inequality."[17]

In India, zero as a concept probably predated zero as a number by hundreds of years. The Sanskrit word for zero, *shunya*, meant "void" or "empty." The word is probably derived from *shuna*, which is the past participle of *svi*, "to grow." In one of the early Vedas, *Rig-veda*, there is another meaning: the sense of "lack" or "deficiency." It is possible that the two different words were fused to give *shunya* a single sense of "absence" or "emptiness" with the potential for growth. Hence, its derivative, *Shunyata*, described the Buddhist doctrine of "Emptiness," being the spiritual practice of emptying the mind of all impressions. This was a course of action prescribed in a wide range of creative endeavors. For example, the practice of *Shunyata* is recommended in writing poetry, composing a piece of music, producing a painting, or in any activity that comes out of the mind of the artist. An architect was advised in the traditional manuals of architecture (the *Silpas*) that designing a building involved the organization of empty space, for "it is not the walls that make a building but the empty spaces created by the walls." The whole process of creation is vividly described in the following verse from a Tantric Buddhist text:

> First the realization of the void [*shunya*],
> Second the seed in which all is concentrated,
> Third the physical manifestation,
> Fourth one should implant the syllable.

The mathematical correspondence was soon established. "Just as emptiness of space is a necessary condition for the appearance of any object, the number zero being no number at all is the condition for the existence of all numbers."

A discussion of the mathematics of the *shunya* involves three related issues: (1) the concept of the *shunya* within a place-value system, (2) the symbols used for *shunya*, and (3) mathematical operations with the *shunya*. Materials from appropriate early texts are used as illustrations below.

It was soon recognized that the *shunya* denoted notational place (place-holder) as well as the "void," or absence of numerical value, in a particular notational place. Consequently all numerical quantities, however great, could be represented with just ten symbols. A twelfth-century text (*Manasollasa*) states:

> Basically, there are only nine digits, starting from "one" and going to "nine." By adding the zeros these are raised successively to tens, hundreds, and beyond.

And in a commentary on Patanjali's *Yogasutra* there appears in the fifth century the following analogy:[18]

> Just as the same sign is called a hundred in the "hundreds" place, ten in the "tens" place, and one in the "units" place, so is one and the same woman referred to (differently) as mother, daughter, or sister.

One of the earliest mentions of a symbol for zero occurs in the *Chandahsutra* of Pingala (fl. third century BC), which discusses a method for calculating the number of arrangements of long and short syllables in a meter containing a certain number of syllables (i.e., the number of combinations of two items from a total of *n* items, repetitions being allowed). The symbol for *shunya* began as a dot (*bindu*), found in inscriptions in India, Cambodia, and Sumatra around the seventh and eighth centuries, and then became a circle (*chidra* or *randhra*, meaning a hole). The association between the concept of zero and its symbol was already well established by the early centuries of the Christian era, as the following quotation shows:

> The stars shone forth, like zero dots [*shunya-bindu*] scattered in the sky as if on a blue rug, [such that] the Creator reckoned the total with a bit of the moon for chalk. (Vasavadatta, c. AD 400)

Sanskrit texts on mathematics/astronomy from the time of Brahmagupta usually contain a section called *shunya-ganita* or computations involving zero. While the discussion in the arithmetical texts (*patiganita*) is limited only to addition, subtraction, and multiplication with zero, the treatment in algebra texts (*bijaganita*) covers such questions as the effect of zero on the positive and negative signs, division with zero, and more particularly the relation between zero and infinity (*ananta*).

Take, as an example, Brahmagupta's seventh-century text *Brahma Sphuta Siddhanta*. In it he treats the zero as a separate entity from the positive (*dhana*) and negative (*rina*) quantities, implying that *shunya* is neither positive nor negative but denotes the boundary between the two kinds, being the sum of two equal but opposite quantities. He states that a number, whether positive or negative, remains unchanged when zero is added to or subtracted from it. In multiplication with zero, the product is zero. A zero divided by zero or by some number becomes zero. Likewise the square and square root of zero is zero. But when a number is divided by zero, the answer is an undefined quantity, "that which has that zero as the denominator."[19] In the twelfth century, Bhaskaracharya stated that if you were to divide by zero you would get a number that was "as infinite as the god Vishnu"!

The Spread of Numeracy in India: A Historical Perspective

A search for the social origins of numeracy must consider the everyday practices and institutions that make the numerals and operations with them familiar to the ordinary person. The structure of Indian mathematics education for all may have been set by a Jaina text, called *Sthananga Sutra*, dating back to about 300 BC. In that, the first two topics out of ten, *parikarma* (number representation and the four fundamental operations of arithmetic) and *vyavahara* (arithmetic problems, including the "rule of three"), came to be referred to as *patiganita* (etymology: "calculation on tablet") and were meant to be studied by all. The other eight topics were plane geometry calculations as carried out with a rope (*rajju*), mensuration of plane figures and solids (*rasi*), advanced treatment of fractions (*kalasavarna*), study of that which is unknown or algebra (*yavat-tavat*), problems involving squares and square roots (*varga*), problems involving cubes and cube roots (*ghana*), problems involving higher powers and higher roots (*varga-varga*), and permutations and combinations (*vikalpa*).

Although being taught at home was the usual practice for the higher-caste males and for all females, all other castes attended schools. There are early British descriptions of indigenous village schools where emphasis on numeracy was an important part of the school curriculum. A report, submitted in 1838 by William Adam, of such schools in certain districts of Bengal and Bihar (Dharampal 1983) is quite illuminating. The period a student spent in an elementary school was divided into four stages. The first stage, when the child first entered school, seldom exceeded ten days. During that time the young child was taught “to form letters of the alphabet on the ground with a small stick or slip of bamboo,” or on a sand board, a board on which sand was sprinkled as a writing surface.The second stage, lasting from two and a half to four years, involved pupils being taught to read from and write on palm leaves. During the same period, the pupil was expected to memorize “the Cowrie Table, the Numeration Table as far as 100, the *Katha* Table and the *Ser* Table,” the latter two being tables of weights and measures. To help them with this enormous task, different systems of word-numerals were taught. The third stage, lasting from two to three years, was spent on improving their literary skills practiced on plantain leaf, as well as completing the basic course on *patiganita.* In the fourth and final stage, lasting up to two years, pupils were expected to read religious and other texts, both at school and at home, undergo training in commercial and agricultural accounts, and compose letters and petitions. A few would continue their education in institutions or within the household, where Sanskrit was the language of instruction and the teachers and students were predominantly Brahmins.

Apart from numeracy skills, *patiganita* consisted of all the mathematics needed for daily living. The *vyavaharaganita* included problems involving calculation of volumes of grains and heaps, estimating amounts in piles of bricks and timber, construction of roads and building, calculation of the time of the day, interest and capital calculations, barter and exchange, and recreational problems. In modern terminology, this was practical mathematics, which included commercial mathematics. The authors who wrote texts on *patiganita*, such as the unknown author of Bakhshali Manuscript, or Mahavira (fl. 850 AD), or Sridhara (fl. AD 800) began with a review of arithmetic operations, though the extent and detail to which this was done varied with different texts; the earlier the text, the more detailed the treatment.

The level of numeracy in traditional Indian society was high, partly because of the manner in which numeracy was acquired and passed on and partly because of the lack of any institutional, religious, or philosophical inhibitions to the acquisition and practice of numeracy. Yet the absence of a commercial revolution in India meant that the social milieu that nurtured interest in matters scientific in Europe was missing. In particular, no artificial language evolved, and while notations were fun and intellectually distracting, they did little to advance science, which ultimately stagnated.

And practical mathematics, the handmaiden of numeracy, continued to remain at the same level for about a thousand years, eventually to be submerged by the rise of Western mathematics. Even the remnants of indigenous numeracy that exist in subterranean occupations, such as astrology and traditional architecture, may soon become a historical memory.

Jaina Mathematics

The rise of Buddhism and Jainism around the middle of the first millennium BC was in part a reaction to some of the excesses of Vedic religious and social practices. The resulting decline in offerings of Vedic sacrifices, which had played such a central role in Hindu ritual, meant that occasions for constructing altars requiring practical skills and geometric knowledge became few and far between. There was also a gradual change in the perception of the role of mathematics: from fulfilling the needs of sacrificial ritual, it became an abstract discipline to be cultivated for its own sake. The Jaina contribution to this change should be recognized. Unfortunately, sources of information on Jaina mathematics are scarce, though there are enough to show how original the work was.

A number of Jaina texts of mathematical importance have yet to be studied, and what we know of them is based almost entirely on later commentaries. Of particular relevance is the old canonical literature: *Surya Prajnapti*, *Jambu Dvipa Prajnapti*, *Sthananga Sutra*, *Uttaradhyayana Sutra*, *Bhagavati Sutra*, and *Anuyoga Dvara Sutra*. The first two works are from the third or fourth century BC, and the others are from at least two centuries later. As mentioned in the previous section, the *Sthananga Sutra* gives a list of mathematical topics that were studied at the time. Expressed in their modern equivalents, they were the theory of numbers, arithmetical operations, geometry, operations with fractions, simple equations, cubic equations, biquadratic (quartic) equations, and permutations and combinations. This classification by the Jains was adopted by later mathematicians.

Given the paucity of existing evidence and the little scrutiny it has received, our survey of Jaina mathematics must be rather piecemeal. We shall examine four main areas in which the Jaina contribution was distinctive.[20]

Theory of Numbers

Like the Vedic mathematicians, the Jains had an interest in the enumeration of very large numbers, which was intimately tied up with their philosophy