Chapter 1

Hippocrates' Quadrature of the Lune

(ca. 440 B.C.)

The Appearance of Demonstrative Mathematics

Our knowledge of the very early development of mathematics is largely speculative, pieced together from archaeological fragments, architectural remains, and educated guesses. Clearly, with the invention of agriculture in the years 15,000–10,000 B.C., humans had to address, in at least a rudimentary fashion, the two most fundamental concepts of mathematics: multiplicity and space. The notion of multiplicity, or "number," would arise when counting sheep or distributing crops; over the centuries, refined and extended by generations of scholars, these ideas evolved into arithmetic and later into algebra. The first farmers likewise would have needed insight into spatial relationships, primarily in regard to the areas of fields and pastures; such insights, carried down through history, became geometry. From the beginnings of civilization, these two great branches of mathematics—arithmetic and geometry—would have coexisted in primitive form.

This coexistence has not always been a harmonious one. A continuing feature of the history of mathematics has been the prevailing tension between the arithmetic and the geometric. There have been times when one branch has overshadowed the other and when one has been regarded as logically superior to its more suspect counterpart. Then a new discovery, a new point of view, would turn the tables. It may come as a surprise that mathematics, like art or music or literature, has been subject to such trends in the course of its long and illustrious history.

We find clear signs of mathematical development in the civilization of ancient Egypt. For the Egyptians, the emphasis was on the practical side of mathematics as a facilitator of trade, agriculture, and the other increasingly complex aspects of everyday life. Archaeological records indicate that by 2000 B.C. the Egyptians had a primitive numeral system as well as some geometric ideas about triangles, pyramids, and the like. There is a tradition, for instance, that Egyptian architects used a clever device for making right angles. They would tie 12 equally long segments of rope into a loop, as shown in Figure 1.1. Stretching five consecutive segments in a straight line from *B* to *C* and then pulling the rope taut at *A*, they thus formed a rigid triangle with a right angle *BAC*. This configuration, laid upon the ground, allowed the workers to construct a perfect right angle at the corner of a pyramid, temple, or other building.

Implicit in this construction is an understanding of the Pythagorean relationship of right triangles. That is, the Egyptians seemed to know that a triangle with sides of length 3, 4, and 5 must contain a right angle. Of course, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, and so we catch an early glimpse of one of the most important relationships in all of mathematics (see Figure 1.2).

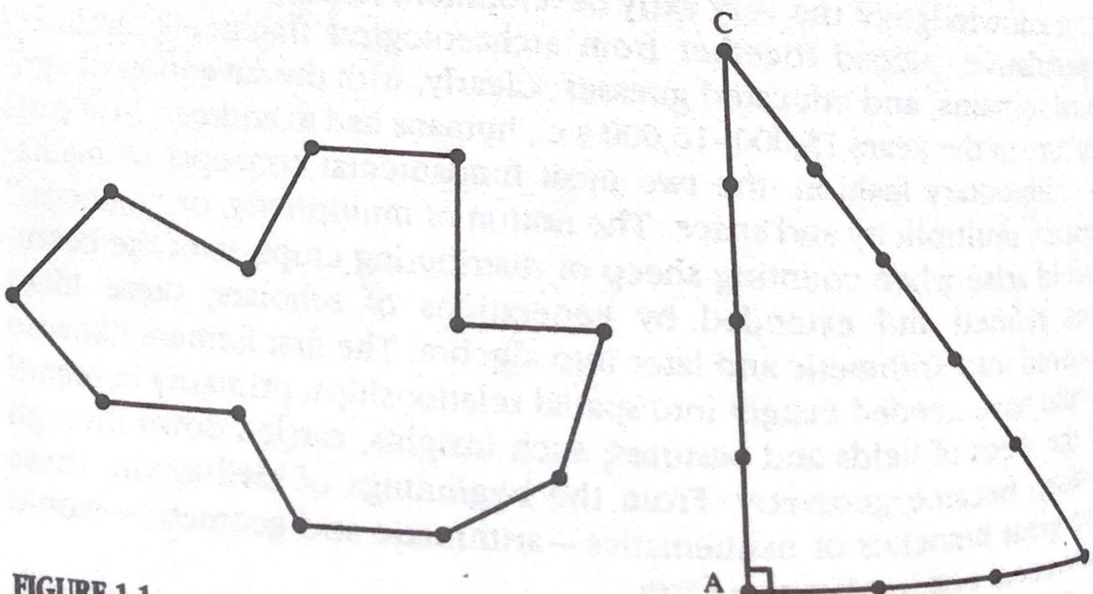

FIGURE 1.1

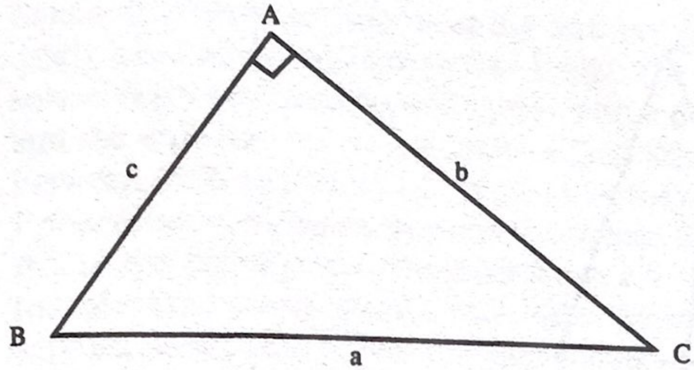

FIGURE 1.2

Technically, this Egyptian insight was not a case of the Pythagorean theorem itself, which states, "If $\triangle BAC$ is a right triangle, then $a^2 = b^2 + c^2$." Rather, it was an example of the *converse* of the Pythagorean theorem: "If $a^2 = b^2 + c^2$, then $\triangle BAC$ is a right triangle." That is, for a proposition of the form "If *P*, then *Q*," the related statement "If *Q*, then *P*" is called the proposition's "converse." As we shall see, a perfectly true statement may have a false converse, but in the case of the famous Pythagorean theorem, both the proposition and its converse are valid. In fact, these will be the "great theorems" in the next chapter.

Although the Egyptians seemed to have some insight into the geometry of 3-4-5 right triangles, it is doubtful they possessed the broader understanding that, for instance, a 5-12-13 triangle or a 65-72-97 triangle likewise contains a right angle (since in each case $a^2 = b^2 + c^2$). More critically, the Egyptians gave no indication of how they might *prove* this relationship. Perhaps they had some logical argument to support their observation about 3-4-5 triangles; perhaps they hit upon it purely by trial and error. In any case, the notion of proving a general mathematical result by a carefully crafted logical argument is nowhere to be found in Egyptian writings.

The following example of Egyptian mathematics may be illuminating: it is their approach to finding the volume of a truncated square pyramid—that is, a square pyramid with its top lopped off by a plane parallel to the base (see Figure 1.3). Such a solid is today called the frustum of a pyramid. The technique for finding its volume appears in the so-called "Moscow Papyrus" from 1850 B.C.:

> If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take a third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

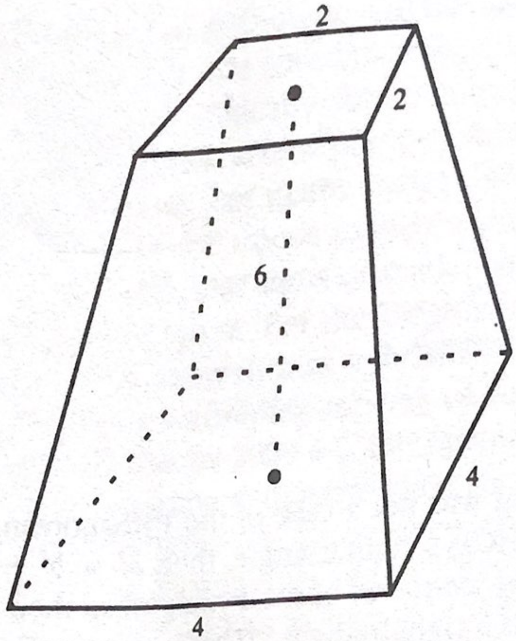

FIGURE 1.3

This is a most remarkable prescription, which indeed yields the correct answer for the frustum's volume. Notice, however, what it does not do. It does not give a general formula to cover frusta of other dimensions. Egyptians would have to generalize from this particular case in order to determine the volume of a different-sized frustum, a process that could be a bit confusing. Far simpler and more concise is our modern formula

$$V = \frac{1}{3}h(a^2 + ab + b^2)$$

where a is the side of the square on the bottom, b is the side of the square on the top, and h is the frustum's height. Worse, there was no indication of *why* this Egyptian recipe provided the correct answer. Instead, a simple "You will find it right" sufficed.

It is probably dangerous to draw sweeping conclusions from a particular example, yet historians have noted that a dogmatic approach to mathematics was certainly in keeping with the authoritarian society that was pharaonic Egypt. Inhabitants of that ancient land were conditioned to give unquestioned obedience to their rulers. By analogy, when presented with an authoritative mathematical technique that concluded "You will find it right," Egyptian subjects were hardly likely to demand a more thorough explanation of why it worked. In the land of the Pharaoh, you did what you were told, whether in erecting a colossal temple or in solving a math problem. Those adamantly questioning the system would end up as mummies before their time.

Another great ancient civilization—or, more precisely, civilizations—flourished in Mesopotamia and produced mathematics significantly more advanced than that of Egypt. The Babylonians, for instance, solved fairly sophisticated problems with a definite algebraic character, and the existence of a clay tablet called Plimpton 322, dated roughly between 1900 and 1600 B.C., shows that they definitely understood the Pythagorean theorem in far more depth than their Egyptian counterparts; that is, the Babylonians recognized that a 5-12-13 triangle or a 65-72-97 triangle (and many more) was right. In addition, they developed a sophisticated place system for their numerals. We, of course, are accustomed to a base-10 numeral system, obviously derived from the 10 fingers of the human hand, so it may seem a bit odd that the Babylonians chose a base-60 system. While no one speculates that these ancient people had 60 fingers, their choice of base can still be seen in our measurement of time (60 seconds per minute) and angles ($6 \times 60° = 360°$ in a circle).

But for all of their achievements, the Mesopotamians likewise addressed only the question of "how" while avoiding the much more significant issue of "why." Those seeking the appearance of a demonstrative mathematics—a theoretical, deductive system in which emphasis was placed upon *proving* critical relationships—would have to look to a later time and a different place.

The time was the first millennium B.C., and the place was the Aegean coasts of Asia Minor and Greece. Here there arose one of the most significant civilizations of history, whose extraordinary achievements would forever influence the course of western culture. Engaged in a thriving commerce, both within their own lands and across the Mediterranean, the Greeks developed into a mobile, adventuresome people, relatively prosperous and sophisticated, and considerably more independent in thought and action than the western world had seen before. These curious, free-thinking merchants were much less likely to submit meekly to authority. Indeed, with the development of Greek democracy, the citizens *became* the authority (although it must be stressed that citizenship in the classical world was very narrowly defined). To such individuals, everything was open to debate and analysis, and ideas were not about to be accepted with a passive, unquestioning obedience.

By 400 B.C., this remarkable civilization could already boast a rich, some would say unsurpassed, intellectual heritage. The epic poet Homer, the historians Herodotus and Thucydides, the dramatists Aeschylus, Sophocles, and Euripides, the politician Pericles, and the philosopher Socrates—these individuals had all left their marks as the fourth century B.C. began. Inhabitants of the modern world, where fame can fade so quickly, may find it astonishing that these names have endured gloriously for over 2000 years. To this day, we admire their boldness in subjecting Nature and the human condition to the penetrating light of reason. Granted, it was reason still contaminated by large doses of superstition and ignorance, but the Greek thinkers were profoundly successful. If their conclusions were not always correct, the Greeks nonetheless sensed that theirs was the path that would lead from a barbarous past to an undreamed-of future. The term "awakening" is often used in describing this special moment in history, and it is apt. Humankind was indeed arising from the slumber of thousands of centuries to confront this strange, mysterious world with Nature's most potent weapon—the human mind.

Such was certainly the case with mathematics. Around 600 B.C. in the town of Miletus on the western coast of Asia Minor, there lived the great Thales (ca. 640–ca. 546 B.C.), one of the so-called "Seven Wise Men" of antiquity. Thales of Miletus is generally credited with being the father of demonstrative mathematics, the first scholar who supplied the "why" along with the "how." As such, he is the earliest known mathematician.

We have very little hard evidence about his life. Indeed, he emerges from the mists of the past as a pseudo-mythical figure, and it is anybody's guess as to the truth of the exploits and discoveries attributed to him. Looking back seven centuries, the biographer Plutarch (A.D. 46–120) wrote that ". . . at that time Thales alone had raised philosophy above mere practice into speculation." A noted mathematician and astronomer who somehow predicted the solar eclipse in 585 B.C., Thales, like the stereotypical scientist, was chronically absent-minded and incessantly preoccupied—according to legend, he once was strolling along, gazing upward at his beloved stars, when he tumbled into an open well.

His "fatherhood" of demonstrative mathematics notwithstanding, Thales never married. When Solon, a contemporary, asked why, Thales arranged a cruel ruse whereby a messenger brought Solon news of his son's death. According to Plutarch, Solon then

> . . . began to beat his head and to do and say all that is usual with men in transports of grief. But Thales took his hand, and, with a smile, said, "These things, Solon, keep me from marriage and rearing children, which are too great for even your constancy to support; however, be not concerned at the report, for it is a fiction."

Clearly, Thales was not the kindest of people. A similar impression emerges from the story of a farmer who routinely tied heavy bags of salt on the back of his donkey when driving the beast to market. The clever animal quickly learned to roll over while fording a particular stream, thereby dissolving much of the salt and making his burden far lighter. Exasperated, the farmer went to Thales for advice, and Thales recommended that on the next trip to market the farmer load the donkey with sponges.

It was certainly not kindness to man or beast that earned Thales his high reputation in mathematics. Rather, it was his insistence that geometric statements not be accepted simply because of their intuitive plausibility; instead they had to be subjected to rigorous, logical proof. This is no small legacy to leave the discipline of mathematics.

What, precisely, are some of his theorems? Tradition holds that it was Thales who first *proved* the following geometric results:

- Vertical angles are equal.
- The angle sum of a triangle equals two right angles.
- The base angles of an isosceles triangle are equal.
- An angle inscribed in a semicircle is a right angle.

In none of these cases do we have any record of his proofs, but we can speculate on their nature. For instance, consider the last proposition above. The proof given below is taken from Euclid's *Elements*, Book III, Proposition 31, but it is simple and direct enough to be a prime candidate for Thales' own.

THEOREM An angle inscribed in a semicircle is a right angle.

PROOF Let a semicircle be drawn with center O and diameter BC, and choose any point A on the semicircle (Figure 1.4). We must prove that $\angle BAC$ is right. Draw line OA and consider $\triangle AOB$. Since OB and OA are radii of the semicircle, they have the same length, and so $\triangle AOB$ is isosceles. Hence, as Thales had previously proved, $\angle ABO$ and $\angle BAO$ are equal (or, in modern terminology, congruent); call them both α. Like-

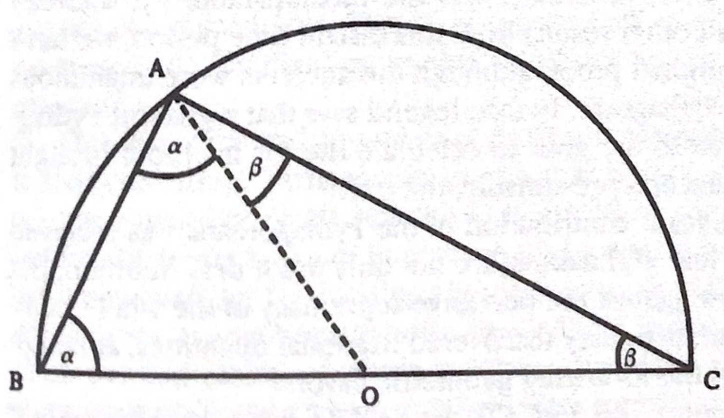

FIGURE 1.4

wise, in $\triangle AOC$, OA and OC have the same length, and so $\angle OAC = \angle OCA$; call them both β. But, from the large triangle BAC, we see that

$$\begin{aligned} \text{2 right angles} &= \angle ABC + \angle ACB + \angle BAC \\ &= \alpha + \beta + (\alpha + \beta) \\ &= 2\alpha + 2\beta = 2(\alpha + \beta) \end{aligned}$$

Hence, one right angle $= \frac{1}{2}[\text{2 right angles}] = \frac{1}{2}[2(\alpha + \beta)] = \alpha + \beta = \angle BAC$. This is exactly what we were to prove.

Q.E.D.

(*Note:* It has become customary, upon the completion of a proof, to insert the letters "Q.E.D.," which abbreviate the Latin *Quod erat demonstrandum* [Which was to be proved]. This alerts the reader to the fact that the argument is over and we are about to set off in new directions.).

After Thales, the next major figure in Greek mathematics was Pythagoras. Born in Samos around 572 B.C., Pythagoras lived and worked in the eastern Aegean, even, according to some legends, studying with the great Thales himself. But when the tyrant Polycrates assumed power in this region, Pythagoras fled to the Greek town of Crotona in southern Italy, where he founded a scholarly society now known as the Pythagorean brotherhood. In their contemplation of the world about them, the Pythagoreans recognized the special role of "whole number" as the critical foundation of all natural phenomena. Whether in music, or astronomy, or philosophy, the central position of "number" was everywhere evident. The modern notion that the physical world can be understood by "mathematization" owes more than a little to this Pythagorean viewpoint.

In the world of mathematics proper, the Pythagoreans gave us two great discoveries. One, of course, was the incomparable Pythagorean theorem. As with all other results from this distant time period, we have no record of the original proof, although the ancients were unanimous in attributing it to Pythagoras. In fact, legend says that a grateful Pythagoras sacrificed an ox to the gods to celebrate the joy his proof brought to all concerned (except, presumably, the ox).

The other significant contribution of the Pythagoreans was received with considerably less enthusiasm, for not only did it defy intuition, but it also struck a blow against the pervasive supremacy of the whole number. In modern parlance, they discovered irrational quantities, although their approach had the following geometric flavor:

Two line segments, AB and CD, are said to be ***commensurable*** if there exists a smaller segment EF that goes evenly into both AB and CD.

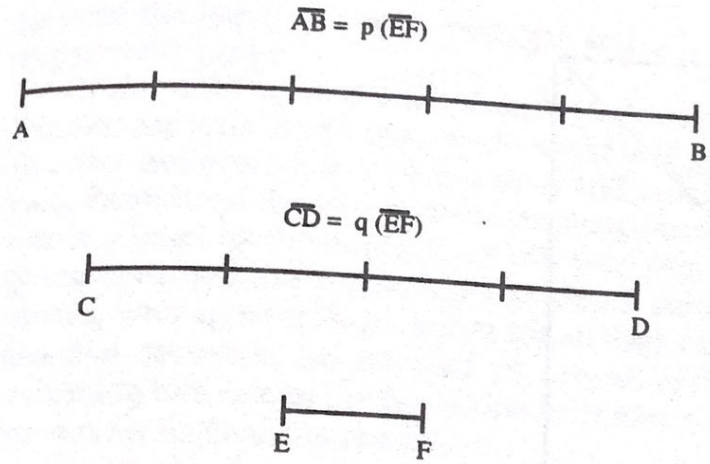

FIGURE 1.5

That is, for some whole numbers p and q, AB is composed of p segments congruent to EF while CD is composed of q such segments (Figure 1.5). Consequently, $\overline{AB}/\overline{CD} = p(\overline{EF})/q(\overline{EF}) = p/q$. (Here we are using the notation $\overline{AB}$ to stand for the length of segment AB). Since p/q is the *ratio* of two positive integers, we say that the ratio of the lengths of commensurable segments is a "rational" number.

Intuitively, the Pythagoreans felt that *any* two magnitudes are commensurable. Given two line segments, it seemed preposterous to doubt the existence of another segment EF dividing evenly into both, even if it took an extremely tiny EF to do the job. The presumed commensurability of segments was critical to the Pythagoreans, not only because they used this idea in their proofs about similar triangles but also because it seemed to support their philosophical stance on the central role of whole numbers.

However, tradition credits the Pythagorean Hippasus with discovering that the side of a square and its diagonal (GH and GI in Figure 1.6) are not commensurable. That is, no matter how small one goes, there is no magnitude EF dividing *evenly* into both the square's side and its diagonal.

This discovery had a number of profound consequences. Obviously, it shattered those Pythagorean proofs that rested upon the supposed commensurability of all segments. It would be almost two centuries before the mathematician Eudoxus found a way to patch up the theory of similar triangles by devising alternative proofs that did not rely upon the concept of commensurability. Secondly, it had an unsettling impact upon the supremacy of whole numbers, for if not all quantities were commensurable, then whole numbers were somehow inadequate to represent the ratios of all geometric lengths. Consequently, the discovery

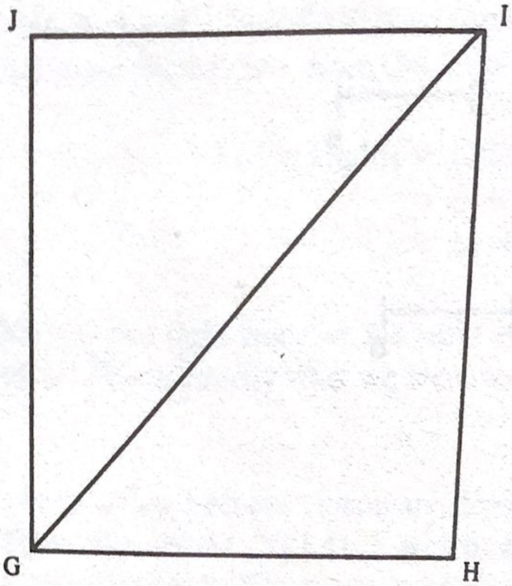

FIGURE 1.6

firmly established the superiority of geometry over arithmetic in all subsequent Greek mathematics. In the Figure 1.6, for instance, the side and diagonal of the square are beyond suspicion as *geometric* objects. But, as *numbers*, they presented a major problem. For, if we imagine that the side of the square above has length 1, then the Pythagorean theorem tells us that the length of the diagonal is $\sqrt{2}$; and, since side and diagonal are not commensurable, we see that $\sqrt{2}$ cannot be written as a rational number of the form p/q. Numerically, then, $\sqrt{2}$ is an "irrational," whose arithmetic character is quite mysterious. Far better, thought the Greeks, to avoid the numerical approach altogether and concentrate on magnitudes simply as geometric entities. This preference for geometry over arithmetic would dominate a thousand years of Greek mathematics.

A final result of the discovery of irrationals was that the Pythagoreans, incensed at all the trouble Hippasus had caused, supposedly took him far out upon the Mediterranean and tossed him overboard to his death. If true, the story indicates the dangers inherent in free thinking, even in the relatively austere discipline of mathematics.

Thales and Pythagoras, while prominent in legend and tradition, are obscure, shadowy figures from the distant past. Our next individual, Hippocrates of Chios (ca. 440 B.C.) is a little more solid. In fact, it is to him that we attribute the earliest mathematical proof that has survived in reasonably authentic form. This will be the subject of our first great theorem.

Hippocrates was born on the island of Chios sometime in the fifth century B.C. This was, of course, the same region that produced his illustrious predecessors mentioned earlier. (Note in passing that Chios is not far from the island of Cos, where another "Hippocrates" was born about this time; it was Hippocrates of Cos—not our Hippocrates—who became the father of Greek medicine and originator of the physicians' Hippocratic oath.)

Of the mathematical Hippocrates, we have scant biographical information. Aristotle wrote that, while a talented geometer, he "... seems in other respects to have been stupid and lacking in sense." This is an early example of the stereotype of the mathematician as being somewhat overwhelmed by the demands of everyday life. Legend has it that Hippocrates earned this reputation after being defrauded of his fortune by pirates, who apparently took him for an easy mark. Needing to make a financial recovery, he traveled to Athens and began teaching, thus becoming him one of the few individuals ever to enter the teaching profession for its financial rewards.

In any case, Hippocrates is remembered for two signal contributions to geometry. One was his composition of the first *Elements,* that is, the first exposition developing the theorems of geometry precisely and logically from a few given axioms or postulates. At least, he is credited with such a work, for nothing remains of it today. Whatever merits his book had were to be eclipsed, over a century later, by the brilliant *Elements* of Euclid, which essentially rendered Hippocrates' writings obsolete. Still, there is reason to believe that Euclid borrowed from his predecessor, and thus we owe much to Hippocrates for his great, if lost, treatise.

The other significant Hippocratean contribution—his quadrature of the lune—fortunately has survived, although admittedly its survival is tenuous and indirect. We do not have Hippocrates' own work, but Eudemus' account of it from around 335 B.C., and even here the situation is murky, because we do not really have Eudemus' account either. Rather, we have a summary by Simplicius from A.D. 530 that discussed the writings of Eudemus, who, in turn, had summarized the work of Hippocrates. The fact that the span between Simplicius and Hippocrates is almost a thousand years—roughly the time between us and Leif Erikson—indicates the immense difficulty historians face when considering the mathematics of the ancients. Nonetheless, there is no reason to doubt the general authenticity of the work in question.

Some Remarks on Quadrature

Before examining Hippocrates' lunes, we need to address the notion of "quadrature." It is obvious that the ancient Greeks were enthralled by the symmetries, the visual beauty, and the subtle logical structure of geometry. Particularly intriguing was the manner in which the simple and elementary could serve as foundation for the complex and intricate. This will become quite apparent in the next chapter as we follow Euclid