correctly pointed out. One readily sees that $x = 2$ is indeed a solution of $x^3 + 6x = 20$.

Further Topics on Solving Equations

Observe that, having found one solution to the cubic, we are now in a position to find any others. For instance, since $x = 2$ solves the specific equation above, we know that $x - 2$ is one factor of $x^3 + 6x - 20$, and long division will generate the other, second-degree factor. In this case, $x^3 + 6x - 20 = (x - 2)(x^2 + 2x + 10)$. The solutions to the original cubic thus arise from solving the linear and quadratic equations

$$x - 2 = 0 \quad \text{and} \quad x^2 + 2x + 10 = 0$$

which is easily done. (This particular quadratic has no real solutions, so the cubic has as its only real solution $x = 2$.)

To the modern reader, the next two chapters of *Ars Magna* seem superfluous. Cardano titled Chapter XII "On the Cube Equal to the First Power and Number"—that is, $x^3 = mx + n$—and Chapter XIII was "On the Cube and Number Equal to the First Power"—that is, $x^3 + n = mx$. Today, we would regard these as having already been adequately covered by the formula above, for we would allow m and n to be negative. Mathematicians in the sixteenth century, however, demanded that all coefficients in the equation be positive. In other words, they regarded $x^3 + 6x = 20$ and $x^3 + 20 = 6x$ not just as different equations, but as intrinsically different ***kinds*** of equations. Such squeamishness about negative numbers is hardly surprising, given Cardano's tendency to think in terms of three-dimensional cubes, where sides of negative length make no sense. Of course, avoiding negatives led to a proliferation of cases and made *Ars Magna* considerably longer than we now find necessary.

So, Cardano could solve the depressed cubic in any of its three versions. But what about the *general* third-degree equation of the form $ax^3 + bx^2 + cx + d = 0$? It was Cardano's great discovery that, by means of a suitable substitution, this equation could be replaced by a related, depressed cubic that was, of course, susceptible to his formula. Before examining this "depressing" process for the cubic, we might take a quick look at it in a more familiar setting—as applied to solving quadratic equations:

Suppose we begin with the general second-degree equation

$$ax^2 + bx + c = 0 \quad \text{where } a \neq 0$$

To depress it—that is, to eliminate its first-power term—we introduce the new variable y by substituting $x = y - b/2a$ to get

$$a\left(y - \frac{b}{2a}\right)^2 + b\left(y - \frac{b}{2a}\right) + c = 0 \quad \text{which gives}$$

$$a\left(y^2 - \frac{b}{a}y + \frac{b^2}{4a^2}\right) + by - \frac{b^2}{2a} + c = 0 \quad \text{or}$$

$$ay^2 - by + \frac{b^2}{4a} + by - \frac{b^2}{2a} + c = 0$$

Then, canceling the by terms, we get the depressed quadratic

$$ay^2 = \frac{b^2}{2a} - \frac{b^2}{4a} - c = \frac{2b^2}{4a} - \frac{b^2}{4a} - \frac{4ac}{4a} = \frac{b^2 - 4ac}{4a}$$

Hence

$$y^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{and} \quad y = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Finally

$$x = y - \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is of course the quadratic formula once again.

As this example suggests, depressing polynomials can prove quite useful. With this in mind, we return to Cardano's attack on the general cubic. Here, the key substitution is $x = y - b/3a$, which yields

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

Upon expanding, this becomes

$$\left(ay^3 - by^2 + \frac{b^2}{3a}y - \frac{b^3}{27a^2}\right) + \left(by^2 - \frac{2b^2}{3a}y + \frac{b^3}{9a^2}\right) + \left(cy - \frac{cb}{3a}\right) + d = 0$$

There is but one critical observation we need to make regarding this blizzard of letters, namely, that the y^2 terms will cancel out. Thus, the new cubic loses its second-degree term (as desired). If we divide through by a, the resulting equation takes the form $y^3 + py = q$. We solve this for y by Cardano's formula and from there have no difficulty in determining $x = y - b/3a$.

To see this process in action, consider the cubic

$$2x^3 - 30x^2 + 162x - 350 = 0$$

With the substitution $x = y - b/3a = y - (-30/6) = y + 5$, we get

$$2(y+5)^3 - 30(y+5)^2 + 162(y+5) - 350 = 0$$

which becomes

$$2y^3 + 12y - 40 = 0 \qquad \text{or simply} \qquad y^3 + 6y = 20$$

But this is, of course, the very depressed cubic we solved earlier, and so we know that $y = 2$. Hence $x = y + 5 = 7$, and this checks in the original equation.

Ars Magna did not handle the general cubic quite so concisely as we did here. Instead, demanding only positive coefficients, Cardano had to wade through a string of different cases, such as "On the Cube, Square, and First Power Equal to the Number," "On the Cube Equal to the Square, First Power, and Number," "On the Cube and Number Equal to the Square and First Power," and so on. At last, 13 chapters after solving the depressed cubic, he brought the matter to its conclusion. The cubic had been solved.

Or had it? Although Cardano's formula seemed to be an amazing triumph, it introduced a major mystery. Consider, for instance, the depressed cubic $x^3 - 15x = 4$.

Using $m = -15$ and $n = 4$ in the formula developed above, we get

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

Obviously, if negative numbers were suspect in the 1500s, their *square roots* seemed absolutely preposterous, and it was easy to dismiss this as an unsolvable cubic. Yet it can easily be checked that the cubic above has three different and perfectly real solutions: $x = 4$ and $x = -2 \pm \sqrt{3}$. What was Cardano to make of such a situation—the so-called "irreducible case of the cubic"? He took a few half-hearted stabs at investigating what we now call "imaginary" or "complex" numbers but ultimately dismissed the whole enterprise as being "as subtle as it is useless."

It would be another generation before Rafael Bombelli (ca. 1526–1573), in his 1572 treatise *Algebra*, took the bold step of regarding imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solutions; that is, while we begin and end in the familiar domain of real numbers, we seem compelled to move into the unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this seemed incredibly strange.

We shall examine briefly what Bombelli did. Temporarily disregarding any latent prejudice against $\sqrt{-1}$, we cube the expression $2 + \sqrt{-1}$ to get

$$\begin{aligned}(2 + \sqrt{-1})^3 &= 8 + 12\sqrt{-1} - 6 - \sqrt{-1}\\ &= 2 + 11\sqrt{-1} = 2 + \sqrt{-121}\end{aligned}$$

But if $(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$, then it surely makes sense to say that

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$

Similarly, we can see that $\sqrt[3]{-2 + \sqrt{-121}} = -2 + \sqrt{-1}$. Then, reexamining the cubic $x^3 - 15x = 4$, Bombelli arrived at the solution

$$\begin{aligned}x &= \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}\\ &= (2 + \sqrt{-1}) - (-2 + \sqrt{-1}) = 4\end{aligned}$$

which is correct!

Admittedly, Bombelli's technique raised more questions than it resolved. For one thing, how does one know beforehand that $2 + \sqrt{-1}$ is going to be the cube root of $2 + \sqrt{-121}$? It would not be until the middle of the eighteenth century that Leonhard Euler could give a sure-fire technique for finding roots of complex numbers. Furthermore, what exactly were these imaginary numbers, and did they behave like their real cousins?

It is true that the full importance of complex numbers did not become evident until the work of Euler, Gauss, and Cauchy more than two centuries later, and we shall meet this topic again in the Epilogue to Chapter 10. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra, and he thereby stands as the last in the line of the great Italian algebraists of the sixteenth century.

One point should be stressed here. Contrary to popular belief, imaginary numbers entered the realm of mathematics not as a tool for solving *quadratics* but as a tool for solving *cubics*. Indeed, mathematicians could easily dismiss $\sqrt{-121}$ when it appeared as a solution to $x^2 + 121 = 0$ (for this equation clearly has no real solutions). But they could not so easily ignore $\sqrt{-121}$ when it played such a pivotal role in yielding the solution $x = 4$ for the previous cubic. So it was cubics, not quadratics, that gave complex numbers their initial impetus and their now-undisputed legitimacy.

We should make a final observation about *Ars Magna*. In Chapter XXXIX, Cardano introduced the solution of the quartic with the words:

> There is another rule, more noble than the preceding. It is Lodovico Ferrari's, who gave it to me on my request. Through it we have all the solutions for equations of the fourth power.

While the procedure is quite complicated, its two key steps should ring a bell:

1. Beginning with a general quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$, depress it using the substitution $x = y - b/4a$ and then divide through by a, to generate a depressed quartic in y:

$$y^4 + my^2 + ny = p$$

2. By cleverly introducing auxiliary variables, replace this quartic by a related *cubic*, which then can be solved using the techniques developed above. Here again, Ferrari invoked the rule-of-thumb that the way to solve an equation of a given degree is to reduce it to the solution of an equation of one degree less.

Those who were capable of reading through this, and all of the other discoveries in *Ars Magna*, must have been breathless by the time they finished. The art of equation solving had been taken to new heights, and Luca Pacioli's original assessment that cubics, let alone quartics, were beyond the reach of algebra had been shattered. It is little wonder that Cardano ended his book with the enthusiastic and rather touching statement: "Written in five years, may it last as many thousands."

Epilogue

One question that the Cardano-Ferrari work left unanswered was the algebraic solution of the quintic, or fifth-degree, equation. Their efforts certainly suggested that such a solution by radicals was possible and even gave an obvious hint as to how to begin. That is, faced with the quintic

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

introduce the transformation $x = y - b/5a$ to depress it to

$$y^5 + my^3 + ny^2 + py + q = 0$$

and then search for some auxiliary variables to reduce this to a quartic equation, which is known to be solvable by radicals. Such an argument was especially appealing not only because it mimicked the approach that had proved so successful in disposing of cubic and quartic equations, but also because, as was well known, any fifth-degree (or, indeed, any odd-degree) polynomial equation *must* have at least one real solution. This follows because the graphs of odd-degree equations look something like that of the specific fifth-degree equation shown in Figure 6.2. That is, they rise ever higher as we move in one direction along the x-axis and fall ever lower as we move in the other direction. Consequently, such functions must be positive somewhere and must be negative somewhere else, and we conclude—using a result technically known as the intermediate value theorem—that the continuous graph must somewhere cross the x-axis. In the diagram of the quintic above, c is such a point, and hence $x = c$ is a solution to $x^5 - 4x^3 - x^2 + 4x - 2 = 0$. A

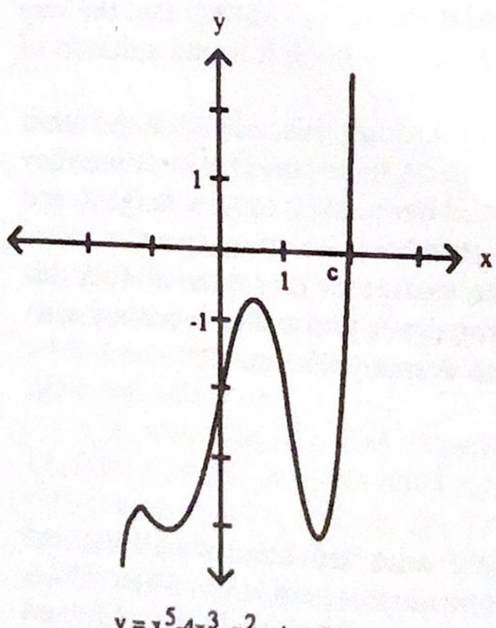

FIGURE 6.2

similar argument guarantees that any odd-degree polynomial equation has (at least) one real solution.

Note, however, that although the intermediate value theorem says that real solutions for quintics *exist*, it by no means gives them explicitly. It was the precise formula for such solutions that the algebraists who followed Ferrari were seeking.

Alas, all efforts in this direction—and they were numerous—met with failure. A century passed, and another, yet no one could provide a "solution by radicals" for the quintic. This came in spite of the fact that later mathematicians found a transformation to reduce the general quintic to one of the form

$$z^5 + pz = q$$

If we called the earlier equation "depressed," this one must have been "utterly despondent." Yet even this highly simplified quintic resisted the efforts of all who attacked it. The situation was frustrating, if not slightly scandalous.

Then, in 1824, a young Norwegian mathematician, Niels Abel (1802–1829), shocked the mathematical world by showing that no "solution by radicals" was possible for fifth- or higher-degree equations. The search, in short, had been doomed from the start. Abel's proof, which can be found in D. E. Smith's *Source Book in Mathematics*, is quite advanced and not at all easy to follow, yet it certainly stands as a landmark in mathematics history.

It is worth noting what Abel's result did and did not imply. He did not say that *no* quintic is solvable, for we obviously can get lucky and solve such equations as $x^5 - 32 = 0$, which clearly has the solution $x = 2$. Further, Abel did not deny that we might solve quintics using techniques other than the algebraic ones of adding, subtracting, multiplying, dividing, and extracting roots. Indeed, the general quintic *can* be solved by introducing entities called "elliptic functions," but these require operations considerably more complicated than those of elementary algebra. In addition, Abel's result did not preclude our approximating solutions for quintic equations as accurately as we—or our computers—wish.

What Abel did do was prove that there exists no algebraic formula, involving only the coefficients of the original quintic equation, that will be a guaranteed generator of solutions. The analogue of the quadratic formula for second-degree equations and Cardano's formula for cubics simply does not exist—it is impossible to provide a universally effective means of finding solutions by radicals for quintics.

The situation is reminiscent of that encountered when trying to square the circle, for in both cases mathematicians are limited by the tools they can employ. For circle squaring, as noted in Chapter 1, the compass and straightedge are simply not powerful enough to get the job done. Likewise, it is the restriction to "solutions by radicals" that hampers mathematicians in their pursuit of the quintic. The familiar operations of algebra are incapable of taming something as wild as a fifth-degree equation.

We seem to be on the brink of a paradox here, for although mathematicians know that quintics must have solutions, Abel showed that there is no algebraic way of finding them. But it is that modifier, algebraic, that keeps us from plunging over the brink into mathematical chaos. Indeed, what Abel actually demonstrated was that algebra does have very definite limits, and for no obvious reason, these limits appear precisely as we move from the fourth to the fifth degree.

Consequently, in a very real sense, we have come full circle. The pessimism of Luca Pacioli, obscured by the thrill of discovery in the sixteenth century, turned out to have been prophetic. When we move beyond fourth-degree equations, the unequivocal triumph of algebra is lost forever.