Voters 1–8	Voters 9–11	Voters 12–13	Voters 14–17
a	*b*	*c*	*d*
b	*c*	*b*	*a*
c	*d*	*d*	*c*
d	*a*	*a*	*b*

and P_2 might be the following profile (involving voters 18–34):

Voters 18–25	Voters 26–28	Voters 29–30	Voters 31–34
a	*c*	*b*	*d*
c	*b*	*c*	*a*
b	*d*	*d*	*b*
d	*a*	*a*	*c*

When P_1 and P_2 are disjoint in this way, $P_1 + P_2$ represents the combined election. Thus, in our example above, $P_1 + P_2$ has 34 voters.

A social choice function is *weakly consistent* if for every pair of disjoint profiles P_1 and P_2 (for the same set of alternatives), if an alternative *x* is among the winners in the P_1 election and among the winners in the P_2 election, then it is also among the winners in the $P_1 + P_2$ election.

(a) Use the above profiles to prove that the Hare system is not weakly consistent.

(b) Using a few sentences, prove that the plurality procedure is weakly consistent.

(c) Using a few sentences, prove that the Borda count is weakly consistent.

J. Smith and H. P. Young used a stronger form of consistency to help characterize an important class of voting rules called scoring rules. The plurality procedure and the Borda count are each scoring rules.

44. Essay Question: Arrow's theorem proves that there is no perfect social choice procedure, but some are certainly better than others. Which social choice procedure do you believe the United States should use to elect its president (within the framework of the Electoral College)?

CHAPTER 2

Yes–No Voting

2.1 INTRODUCTION

In Chapter 1, we considered voting systems in which the voters were choosing among several candidates or alternatives. In the present chapter, we deal with quite a different voting situation—the one in which a single alternative, such as a bill or an amendment, is pitted against the status quo. In theses systems each voter responds with a vote of "yea" or "nay." A *yes–no voting system* is simply a set of rules that specifies exactly which collections of "yea" votes yield passage of the issue at hand.

We begin in **Section 2.2** with four real-world examples of yes–no voting systems, including the all-important U. S. federal system. In **Section 2.3** we introduce the notion of a "weighted system." These are important because they are the ones that we feel we understand (in some sense) the best. We show that—surprisingly—the voting system used in the U. N. Security Council is a weighted system. This result suggests the tantalizing possibility that *all* yes–no voting systems are weighted.

In **Section 2.4** we show that, alas, the U. S. federal system is *not* a weighted voting system. We do this via a notion known as swap robustness. We show that every weighted system is swap robust, but the U. S. federal system is not swap robust. In **Section 2.5** we generalize the notion of swap robustness to something called trade robustness, and we show that the 1982 procedure to amend the Canadian constitution is swap robust but not trade robust. We conclude in **Section 2.6** by stating the characterization theorem asserting that trade robustness and weightedness are fully equivalent.

For a monograph-length treatment of yes–no voting systems, see the 1999 book *Simple Games* by Alan Taylor and William Zwicker.

2.2 FOUR EXAMPLES OF YES–NO VOTING SYSTEMS

Example 1: The European Economic Community

In 1958, the Treaty of Rome established the existence of a yes–no voting system called the European Economic Community. The voters in this system were the following six countries:

France
Germany
Italy
Belgium
the Netherlands
Luxembourg.

France, Germany, and Italy were given four votes each, while Belgium and the Netherlands were given two votes and Luxembourg one. Passage required a total of at least twelve of the seventeen votes. The European Economic Community was altered in 1973 with the addition of new countries and a reallocation of votes. This version of the European Economic Community is discussed later.

Example 2: The United Nations Security Council

The voters in this system are the fifteen countries that make up the Security Council, five of which (China, England, France, Russia, and the United States) are called permanent members whereas the other

ten are called nonpermanent members. Passage requires a total of at least nine of the fifteen possible votes, subject to a veto due to a nay vote from any one of the five permanent members. (For simplicity, we ignore the possibility of abstentions.)

Example 3: The United States Federal System

There are 537 voters in this yes–no voting system: 435 members of the House of Representatives, 100 members of the Senate, the vice president, and the president. The vice president plays the role of tiebreaker in the Senate, and the president has veto power that can be overridden by a two-thirds vote of both the House and the Senate. Thus, for a bill to pass it must be supported by either:

1. 218 or more representatives and 51 or more senators (with or without the vice president) and the president.
2. 218 or more representatives and 50 senators and the vice president and the president.
3. 290 or more representatives and 67 or more senators (with or without either the vice president or the president).

Example 4: The System to Amend the Canadian Constitution

Since 1982, an amendment to the Canadian Constitution becomes law only if it is approved by at least seven of the ten Canadian provinces subject to the proviso that the approving provinces have, among them, at least half of Canada's population. For our purposes, it will suffice to work with the following population percentages for the ten Canadian provinces, based on *Statistics Canada* estimates as of January 1, 2007 (rounded):

Prince Edward Island (0%)

Newfoundland (2%)

New Brunswick (2%)

Nova Scotia (3%)

Saskatchewan (3%)

Manitoba (4%)

Alberta (11%)

British Columbia (13%)

Quebec (23%)

Ontario (39%)

In a yes–no voting system, any collection of voters is called a *coalition*. A coalition is said to be *winning* if passage is guaranteed by yes votes from exactly the voters in that coalition. Coalitions that are not winning are called *losing*. Thus, every coalition is either winning or losing. In Example 1, the coalition made up of France, Germany, and Italy is a winning coalition, as is the coalition made up of France, Germany, Italy, and Belgium. Note that when one asserts that a collection of voters is a winning coalition, nothing is being said about *how* these players actually voted on a particular issue. One is simply saying that *if* these people voted for passage of some bill and the other players voted against passage of that bill, the bill would, in fact, pass.

For most yes–no voting systems, adding extra voters to a winning coalition again yields a winning coalition. Systems with this property are said to be *monotone*. For monotone systems, one can concentrate on the so-called *minimal winning coalitions*: those winning coalitions with the property that the deletion of one or more voters from the coalition yields a losing coalition. In our example above, France, Germany, and Italy make up a minimal winning coalition, while France, Germany, Italy, and Belgium do not. (In the European Economic Community, the minimal winning coalitions are precisely the ones with exactly twelve votes—see Exercise 2.)

2.3 WEIGHTED VOTING AND THE U.N. SECURITY COUNCIL

The four examples of yes–no voting systems in the last section suggest there are at least three distinct ways in which a yes–no voting system can be described:

1. One can specify the number of votes each player has and how many votes are needed for passage. This is what was done for the European Economic Community. More generally, if we start with a set of voters, then we can construct a yes–no voting system by assigning real number *weights* to the voters (allowing for what we might think of as either a fractional number of votes for some voter, or even a negative number of votes) and then set any real number q as the "quota." A coalition is then declared to be winning precisely when the sum of the weights of the voters who vote "yea" meets or exceeds the quota. (Even in the real world, quotas can be less than half the sum of the weights—see Exercise 1.)

2. One can explicitly list the winning coalitions, or, if the system is monotone, just the minimal winning coalitions. This is essentially what is done in the description of the U.S. federal system above, since the three clauses given there describe the three kinds of winning coalitions in the U.S. federal system. In fact, if one deletes the parenthetical clauses and the phrase "or more" from those descriptions, the result is a description of the three kinds of minimal winning coalitions in the U.S. federal system.

3. One can use some combination of the above two, with provisos that often involve veto power. Both the U.N. Security Council and the procedure to amend the Canadian Constitution are described in this way. Moreover, the description of the U.S. federal system in terms of the tie-breaking vote of the vice president, the presidential veto, and the Congressional override of this veto is another example of a description mixing weights with provisos and vetoes. (We say "weights" in this context since,