26. Consider the 6-person voting system in which voters A, B, and C belong to chamber 1, and voters D, E, and F belong to chamber 2. Suppose that a coalition is winning precisely when it contains at least 2 voters from each chamber. Prove that this system is not swap robust.
27. Suppose we have a 7-person yes–no voting system with a 4-person House $H = \{a, b, c, d\}$ and a 3-person Senate $S = \{x, y, z\}$. Suppose that a coalition is winning when it has a total of at least 3 voters, at least one of which is from the Senate.
    (a) Prove or disprove that this system is swap robust.
    (b) Prove or disprove that this system is trade robust.
28. Suppose that a yes–no voting system is swap robust, and that we create a new yes–no voting system by giving voter p veto power (thus, a coalition is winning in the new system precisely when it is both winning in the old system and contains the voter p).
    (a) Prove that the new system is swap robust.
    (b) If we alter a yes–no voting system by giving 3 voters veto power, is it still swap robust (why or why not)?
29. Use trade-robustness to prove that if we have a weighted yes–no voting system, and we create a new system by giving some of the voters veto power, then the resulting system is still weighted.

CHAPTER

3

Political Power

3.1 INTRODUCTION

One of the central concepts of political science is power. While power itself is certainly many-faceted (with aspects such as influence and intimidation coming to mind), our concern is with the narrower domain involving power as it is reflected in formal voting situations (most often) related to specific yes–no issues. If everyone has one vote and majority rule is being used, then clearly everyone has the same amount of "power." Intuition might suggest that if one voter has three times as many votes as another (and majority rule is still being used in the sense of "majority of votes" being needed for passage), then the former has three times as much power as the latter. The following hypothetical example should suffice to call this intuition (or this use of the word *power*) into question.

Suppose the United States approaches its neighbors Mexico and Canada with the idea of forming a three-member group analogous to the European Economic Community as set up by the Treaty of Rome in 1958. Recall that France, Germany, and Italy were given four votes each, Belgium and the Netherlands two each, and Luxembourg one vote, for a total of seventeen votes, with twelve of the seventeen votes needed for passage. Now suppose that in our hypothetical example we suggest mimicking this with the United States getting three votes while each of its two smaller neighbors gets one vote. With this total of five votes we could also suggest using majority rule (three or more out of five votes) for passage and argue that it is not unreasonable for the United States to have three times as much "power" as either Canada or Mexico. In this situation it is certainly unlikely that either Canada or Mexico would be willing to go along with the previously suggested intuition aligning "three times as many votes" with "three times as much power."

In the hypothetical example above, Canada and Mexico have no "power" (although they have votes). So what is this aspect of power that they are completely without? As an answer, "control over outcomes" suggests itself, and, indeed, much of the present chapter is devoted to quantitative measures of power that directly incorporate this control-over-outcomes aspect of power. (It also turns out—and we'll discuss this in more detail later—that these quantitative measures of power indicate that Luxembourg fared no better in the original European Economic Community of 1958 than would Canada and Mexico in our hypothetical example.)

In **Section 3.2** we consider the most well-known cardinal notion of power: the Shapley–Shubik index. This notion of power applies to any yes–no voting system (and not just to weighted voting systems). The mathematical preliminaries involved here include the "multiplication principle" and its corollary giving the number of distinct arrangements of n objects. In **Section 3.3**, we calculate the Shapley–Shubik indicies for the European Economic Community, and we use later developments in this voting system to illustrate a phenomenon known as the "paradox of new members" (where one's power is actually increased in a situation where it appears to have been diluted).

Section 3.4 contains the second most well-known cardinal notion of power: the Banzhaf index. In **Section 3.5**, we introduce two methods, dating back to Allingham (1975) and Dahl (1957), for calculating the Banzhaf index, and we illustrate these methods using both the European Economic Community and a new paradox of Felsenthal and Machover (1994) wherein a voter's power, as measured by the Banzhaf index, increases after giving away a vote. (A paradox, due to William

Zwicker, that applies to the Shapley–Shubik index but not the Banzhaf index is given in the exercises.)

Finally, in **Section 3.6** we present an aspect of political power that is quite different from those considered earlier in the chapter. The issue here is a well-known paradox called the Chair's paradox, and we use it to illustrate that naïve measures of power need not correspond to influence over outcomes.

Before continuing, we add one convention that will be in place throughout this chapter (and Chapter 9):

> **CONVENTION.** Whenever we say "voting system" we mean "monotone voting system in which the grand coalition (the one to which all the voters belong) is winning, and the empty coalition (the one to which none of the voters belong) is losing."

With this convention at hand, we can now turn to our discussion of power.

3.2 THE SHAPLEY-SHUBIK INDEX OF POWER

We begin with some mathematical preliminaries. Suppose we have n people $p_1, p_2, \ldots, p_n$ where n is some positive integer. In how many different ways (i.e., orders) can we arrange them? We check it for some small values of n in Figure 1.

Notice how the orderings for $n = 2$ in Figure 1 arise from the single ordering p_1 for $n = 1$; that is, p_2 can be placed in either the "box" before the p_1 or the "box" after the p_1, as illustrated in Figure 2.

$n = 1$: clearly only one way	p_1
$n = 2$: two ways	p_2p_1 and p_1p_2
$n = 3$: six ways	$p_3p_2p_1$; $p_2p_3p_1$; $p_2p_1p_3$
	and
	$p_3p_1p_2$; $p_1p_3p_2$; $p_1p_2p_3$

FIGURE 1

□ p_1 and p_1 □

FIGURE 2

□ $p_2 p_1$; p_2 □ p_1; $p_2 p_1$ □ and □ $p_1 p_2$; p_1 □ p_2; $p_1 p_2$ □

FIGURE 3

Closer analysis reveals that the same thing is happening as we go from the $n = 2$ case to the $n = 3$ case. That is, each of the two orderings of p_1 and p_2 gives rise to three orderings of p_1, p_2, and p_3 depending upon in which of the three boxes we choose to place p_3. This is illustrated in Figure 3.

If we were to display the $n = 4$ case as in Figure 1, then it should be clear that for each of the six orderings of p_1, p_2 and p_3 we'd have four boxes in which to place p_4. With four orderings so arising from each of the previous six, we would see a total of twenty-four. Examining the sequence of numbers that are suggesting themselves reveals the following:

If $n = 1$, the number of orderings is 1.
If $n = 2$, the number of orderings is $2 = 2 \times 1$.
If $n = 3$, the number of orderings is $6 = 3 \times 2 = 3 \times 2 \times 1$.
If $n = 4$, the number of orderings is $24 = 4 \times 6 = 4 \times 3 \times 2 = 4 \times 3 \times 2 \times 1$.

In general, the number of different ways that n people can be arranged (i.e., ordered) is $(n) \times (n-1) \times (n-2) \times \ldots \times (3) \times (2) \times (1)$. This number is called "n factorial" and is denoted by "$n!$" (e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$). All of this is formalized in the following three results.

PROPOSITION 1 (*The Multiplication Principle*). *Suppose we are considering objects each of which can be built in two steps. Suppose there are exactly f (for "first") ways to do the first step and exactly s (for "second") ways to do the second step. Then the number of such objects (that can be built altogether) is the product $f \times s$. (We are assuming that different construction scenarios produce different objects.)*

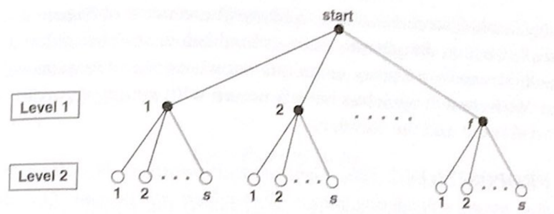

FIGURE 4

PROOF. Consider Figure 4, where the dots (from now on called *nodes*) labeled $1, 2, \ldots, f$ on the first level represent the f ways to do the first step in the construction process, and, for each of these, the nodes labeled $1, 2, \ldots, s$ represent the s ways to do the second step.

Notice that each node on level 2 (the so-called *terminal nodes*) corresponds to a two-step construction scenario. Moreover, the number of terminal nodes is clearly $f \times s$ since we have f "clumps" (one for each node on level 1) and each "clump" is of size s. This completes the proof.

Suppose now that we are building objects by a three-step process where there are k_1 ways to do the first step, k_2 ways to do the second, and k_3 ways to do the third step. How many such objects can be constructed? The answer, it turns out, can be derived from Proposition 1 because we can regard this three-step process as taking place in two "new steps" as follows:

1. New step one: same as old step one.
2. New step two: do the old step two and then the old step three.

Notice that Proposition 1 tells us there are $k_2 \times k_3$ new step twos. Since we know there are k_1 new step ones, we can apply Proposition 1 again to conclude that the number of objects built by our new two-step process (equivalently, by our old three-step process) is given by:

$$k_1 \times (k_2 \times k_3) = k_1 \times k_2 \times k_3.$$

One could also derive this by looking at a version of Figure 4 with three levels, and the general result—stated below as Proposition 2—is usually derived via a proof technique known as mathematical induction. We'll content ourselves here, however, with simply recording the general result and the corollary.

PROPOSITION 2 *(The General Multiplication Principle). Suppose we are considering objects all of which can be built in n steps. Suppose there are exactly k_1 ways to do the first step, k_2 ways to do the second step, and so on up to k_n ways to do the n^{th} step. Then the number of such objects (that can be built altogether) is*

$$k_1 \times k_2 \times \ldots \times k_n,$$

assuming that different construction scenarios produce different objects.

As an application of Proposition 2, suppose we have n people and the objects we are building are arrangements (i.e., orders) of the people. Each ordering can be described as taking place in n steps as follows:

Step 1: Choose one person (from the n) to be first.

Step 2: Choose one person (from the remaining $n - 1$) to be second.

⋮

Step $n - 1$: Choose one person (from the remaining 2) to be $n - 1^{\text{st}}$.

Step n: Choose the only remaining person to be last.

Clearly there are n ways to do step 1, $n - 1$ ways to do step 2, $n - 2$ ways to do step 3, and so on down to 2 ways to do step $n - 1$ and 1 way to do step n. Thus, an immediate corollary of Proposition 2 is the following:

COROLLARY. *The number of different ways that n people can be arranged is*

$$n \times (n-1) \times (n-2) \times \cdots \times (3) \times (2) \times (1),$$

which is, of course, just $n!$ (factorial, not surprise).

One final idea—that of a "pivotal player"—is needed before we can present the formal definition of the Shapley–Shubik index. Suppose, for example, that we have a yes–no voting system with seven players: $p_1, p_2, p_3, p_4, p_5, p_6, p_7$. Fix one of the 7! orderings; for example, let's consider:

$$p_3\ p_5\ p_1\ p_6\ p_7\ p_4\ p_2.$$

We want to identify one of the players as being "pivotal" for this ordering. To explain this idea, we picture a larger and larger coalition being formed as we move from left to right. That is, we first have p_3 alone, then p_5 joins to give us the two-member coalition p_3, p_5. Then p_1 joins, yielding the three-member coalition p_3, p_5, p_1. And so on. The pivotal person for this ordering is the one whose joining converts this growing coalition from a non-winning one to a winning one. Since the empty coalition is losing and the grand coalition is winning (by our convention in **Section 3.1**), it is easy to see that some voter must be pivotal.

Example:

Suppose $X = \{p_1, \ldots, p_7\}$ and each player has one vote except p_4 who has three. Suppose five votes are needed for passage. Consider the ordering: $p_7p_3p_5p_4p_2p_1p_6$. Then, since $\{p_7, p_3, p_5\}$ is not a winning coalition, but $\{p_7, p_3, p_5, p_4\}$ is a winning coalition, we have that the pivotal player for this ordering is p_4.

The Shapley–Shubik index of a player p is the number between zero and one that represents the fraction of orderings for which p is the pivotal player. Thus, being pivotal for lots of different orderings corresponds to having a lot of power according to this particular way of measuring power. More formally, the definition runs as follows.

DEFINITION. Suppose p is a voter in a yes–no voting system and let X be the set of all voters. Then the *Shapley–Shubik index* of p, denoted here by $\text{SSI}(p)$, is the number given by:

$$\text{SSI}(p) = \frac{\text{the number of orderings of } X \text{ for which } p \text{ is pivotal}}{\text{the total number of possible orderings of the set } X}.$$

Note the following:

1. The denominator in SSI(p) is just $n!$ if there are n voters.
2. For every voter p we have $0 \leq \text{SSI}(p) \leq 1$.
3. If the voters are $p_1, \ldots, p_n$, then $\text{SSI}(p_1) + \ldots + \text{SSI}(p_n) = 1$.

Intuitively, think of SSI (p) as the "fraction of power" that p has. The following easy example is taken from Brams (1975); it is somewhat striking.

Example:

Suppose we have a three-person weighted voting system in which p_1 has fifty votes, p_2 has forty-nine votes, and p_3 has one vote. Assume fifty-one votes are needed for passage. The six possible orderings ($3! = 3 \times 2 \times 1 = 6$) are listed below, and the pivotal player for each has been circled.

p_1	(p_2)	p_3
p_1	(p_3)	p_2
p_2	(p_1)	p_3
p_2	p_3	(p_1)
p_3	(p_1)	p_2
p_3	p_2	(p_1)

Since p_1 is pivotal in four of the orderings, $\text{SSI}(p_1) = \frac{4}{6} = \frac{2}{3}$.
Since p_2 is pivotal in one of the orderings, $\text{SSI}(p_2) = \frac{1}{6}$.
Since p_3 is pivotal in one of the orderings, $\text{SSI}(p_3) = \frac{1}{6}$.
Notice that although p_2 has forty-nine times as many votes as p_3, they each have the same fraction of power (at least according to this particular way of measuring power).

3.3 CALCULATIONS FOR THE EUROPEAN ECONOMIC COMMUNITY

We now return to the European Economic Community as set up in 1958 and calculate the Shapley–Shubik index for the member countries. Recall that France, Germany, and Italy had four votes, Belgium and the Netherlands had two votes, and Luxembourg had one vote. Passage required at least twelve of the seventeen votes.

Let's begin by calculating SSI(France). We'll need to determine how many of the $6! = 720$ different orderings of the six countries have France as the pivotal player. Because 720 is a fairly large number, we will want to get things organized in such a way that we can avoid looking at the 720 orderings one at a time.

Notice first that France is pivotal for an ordering precisely when the number of votes held by the countries to the left of it is either eight, nine, ten, or eleven. (If the number were seven or less, then the addition of France's four votes would yield a total of at most eleven, and thus not make it a winning coalition. If the number were twelve or more, it would be a winning coalition without the addition of France.) We'll handle these four cases separately, and then just add together the number of orderings from each case in which France is pivotal to get the desired final result.

Case 1: Exactly Eight Votes Precede France

There are three ways to total eight with the remaining numbers. We'll handle each of these as a subcase.

1.1: France is Preceded by Germany, Belgium, and the Netherlands (with Votes 4, 2, and 2)

In this subcase, the three countries preceding France can be ordered in $3! = 6$ ways, and for each of these six, the two countries following France (Italy and Luxembourg in this case) can be ordered in $2! = 2$ ways. Thus we have $6 \times 2 = 12$ distinct orderings in this subcase. (Equivalently, the number of orderings in this subcase—by Proposition 2—is $3 \times 2 \times 1 \times 1 \times 2 \times 1 = 12$.)

1.2: France is Preceded by Italy, Belgium, and the Netherlands (with Votes 4, 2, and 2)

This case is exactly as 1.1, since both Germany and Italy have four votes.

1.3: France is Preceded by Germany and Italy (with Votes 4 and 4)

In this subcase, the two countries preceding France can be ordered in $2! = 2$ ways, and for each of these two, the three countries following France (Belgium, the Netherlands, and Luxembourg in this case) can be ordered in $3! = 6$ ways. Thus we have $2 \times 6 = 12$ distinct orderings in this subcase, also.

Hence, in case 1 we have a total of 36 distinct orderings in which France is pivotal. For the next three cases (and their subcases), we'll leave the calculations to the reader and just record the results.

Case 2: Exactly Nine Votes Precede France

2.1: France is Preceded by Germany, Belgium, the Netherlands, and Luxembourg (with Votes 4, 2, 2, and 1)

The number of orderings here turns out to be $4! \times 1! = 24 \times 1 = 24$.

2.2: France is Preceded by Italy, Belgium, the Netherlands, and Luxembourg (with Votes 4, 2, 2, and 1)

As in 2.1, the number of orderings here is 24.

2.3: France is Preceded by Germany, Italy, and Luxembourg (with Votes 4, 4, and 1)

The number of orderings turns out to be $3! \times 2! = 6 \times 2 = 12$.

Hence, in case 2 we have a total of 60 distinct orderings in which France is pivotal.

Case 3: Exactly Ten Votes Precede France

3.1: France is Preceded by Germany, Italy, and Belgium (with Votes 4, 4, and 2)

The number of orderings turns out to be $3! \times 2! = 6 \times 2 = 12$.

3.2: France is Preceded by Germany, Italy, and the Netherlands (with Votes 4, 4, and 2)

Exactly as in 3.1, the number here is 12.

Hence, in case 3 we have a total of 24 distinct orderings in which France is pivotal.

Case 4: Exactly Eleven Votes Precede France

4.1: France is Preceded by Germany, Italy, Belgium, and Luxembourg (with Votes 4, 4, 2, and 1)

The number of orderings turns out to be $4! \times 1! = 24 \times 1 = 24$.

4.2: France is Preceded by Germany, Italy, the Netherlands, and Luxembourg (with Votes 4, 4, 2, and 1)

Exactly as in 4.1, the number here is 24.

Hence, in case 4 we have a total of 48 distinct orderings in which France is pivotal.

Finally, to calculate the Shapley–Shubik index of France, we simply add up the number of orderings from the above four cases (giving us the number of orderings for which France is pivotal), and divide by the number of distinct ways of ordering six countries (which is $6! = 720$). Thus,

$$\text{SSI(France)} = \frac{36 + 60 + 24 + 48}{720} = \frac{168}{720} = \frac{14}{60} \approx 23.3\%$$

Germany and Italy also have a Shapley–Shubik index of 14/60 since, like France, they have four votes. It turns out that the Netherlands and Belgium both have a Shapley–Shubik index of 9/60, although we'll leave this as an exercise (which can be done in two different ways) at the end of the chapter. Another exercise is to show that poor Luxembourg has a Shapley–Shubik index of zero! (Hint: in order for Luxembourg to be pivotal in an ordering, exactly how many votes would have to be represented by countries preceding Luxembourg in the ordering? What property of the numbers giving the votes for the other five countries makes this total impossible?) These results are summarized in the following chart:

Country	Votes	Percentage of votes	SSI	Percentage of power
France	4	23.5	14/60	23.3
Germany	4	23.5	14/60	23.3
Italy	4	23.5	14/60	23.3
Belgium	2	11.8	9/60	15.0
Netherlands	2	11.8	9/60	15.0
Luxembourg	1	5.9	0	0

We conclude this section by using the European Economic Community to illustrate a well-known paradox that arises with cardinal notions of power such as those considered in the present chapter (and later in Chapter 9). The setting is as follows: Suppose we have a weighted voting body as set up among France, Germany, Italy, Belgium, the Netherlands, and Luxembourg in 1958. Suppose now that new members are added and given votes, but the percentage of votes needed for passage remains about the same. Intuitively, one would expect the "power" of the original players to become somewhat diluted, or, at worst, to stay the same. The rather striking fact that this need not be the case is known as the "Paradox of New Members." It is, in fact, precisely what occurred when the European Economic Community expanded in 1973.

Recall that in the original European Economic Community, France, Germany, and Italy each had four votes, Belgium and the Netherlands each had two votes, and Luxembourg had one, for a total of seventeen. Passage required twelve votes, which is 70.6 percent of the seventeen available votes. In 1973, the European Economic Community was expanded by the addition of England, Denmark, and Ireland. It was decided that England should have the same number of votes as France, Germany, and Italy, but that Denmark and Ireland should have more votes than the one held by Luxembourg and fewer than the two held by Belgium and the Netherlands. Thus, votes for the original members were scaled up by a factor of $2\frac{1}{2}$, except for Luxembourg, which only had its total doubled. In summary then, the countries and votes stood as follows:

France	10	Belgium	5	England	10
Germany	10	Netherlands	5	Denmark	3
Italy	10	Luxembourg	2	Ireland	3

The number of votes needed for passage was set at forty-one, which is 70.7 percent of the fifty-eight available votes.

The striking thing to notice is that Luxembourg's power—as measured by the Shapley–Shubik index—has increased. That is, while Luxembourg's Shapley–Shubik index had previously been zero, it is clearly greater than zero now since we can produce at least one ordering of the nine countries for which Luxembourg is pivotal. (The actual production of such an ordering is left as an exercise at the end of the chapter.) Notice also that this increase of power is occurring in spite of the fact that Luxembourg was treated worse than the other countries in the scaling-up process. For some even more striking instances of this paradox of new members phenomenon, see the exercises at the end of the chapter where, for example, it is pointed out that even if Luxembourg had been left with one vote, its power still would have increased.

3.4 THE BANZHAF INDEX OF POWER

A measure of power that is similar to (but not the same as) the Shapley–Shubik index is the so-called Banzhaf index of a player. This power index was introduced by the attorney John F. Banzhaf III in connection with a lawsuit involving the county board of Nassau County, New York in the 1960s (see Banzhaf, 1965). The definition takes place via the intermediate notion of what we shall call the "total Banzhaf power" of a player. The definition follows.

DEFINITION. Suppose that p is a voter in a yes–no voting system. Then the total Banzhaf power of p, denoted here by TBP(p), is the number of coalitions C satisfying the following three conditions:

1. p is a member of C.
2. C is a winning coalition.
3. If p is deleted from C, the resulting coalition is not a winning one.

If C is a winning coalition, but the coalition resulting from p's deletion from C is not, then we say that *p's defection from C is critical.*