CHAPTER

5

Fairness

5.1 INTRODUCTION

The central difficulty in solving most disputes is finding a solution that all parties involved consider "fair." Of course, fairness is a subjective issue, and is very difficult to define or quantify. Surprisingly, however, a mathematical perspective can help identify what it means for a solution to be fair and offer a variety methods or procedures for achieving a solution in many types of disputes.

We will focus primarily on fairness in two distinct realms: apportionment and fair division. According to the United States Constitution, the number of congressional representatives per state should be assigned according to the state's population. A naïve allocation of seats to states—based on the fraction of the U.S. population residing in that state—leads to an allocation in which the number of congressional representatives for a state is not a whole number; such an allocation is impossible to implement. And yet we can't just round to the nearest whole number because the sum of the seats allocated has to be a certain fixed number (435). There are, in point of fact, a number of procedures for handling this "rounding-off problem" that have been proposed and used over the years, and we consider several in **Sections 5.2** and **5.3**.

Just as the search for a perfect social choice procedure proved doomed in Chapter 1, so too does the search for a perfect method of apportionment. We illustrate this in **Section 5.4** with a weakened-but-still-striking version of the Balinski-Young impossibility theorem.

Another common type of dispute involves fairly allocating a number of goods among several parties, such as that which occurs in the distribution of marital assets in a divorce or the division of an estate among two or more heirs. In fact, similar methods can be used for disputes in which it is not a physical set of goods that is under contention but a set of issues that need to be resolved. For example, two political parties deciding on rules for a debate between candidates might argue over the issues of the length of the debate, the source of the questions, and the time allowed for initial answers and rebuttals. We will look at what it means for solutions to these types of disputes to be fair, and how to achieve such a solution.

In **Section 5.5**, we discuss fairness in the context of dispute resolution, and some criteria by which we can judge the "fairness" of a particular method of dispute resolution or fair division. In **Section 5.6**, we look at a specific method, the adjusted winner procedure, for disputes involving two parties. In **Section 5.7** we apply the adjusted winner procedure to the Israeli - Palestinian conflict in the Middle East.

5.2 THE PROBLEM OF APPORTIONMENT

The U.S. House of Representatives has, at any given time, a fixed size—presently 435. Article 1, Section 2 of the Constitution specifies that these seats should be apportioned among the states "according to their respective numbers." This suggests that a state with 10% of the U.S. population should have 10% of the 435 seats in the House. Alas, 10% of 435 is 43.5, and a fraction of a seat is quite impossible.

A number such as 43.5, arrived at as we did in the previous paragraph, is a state's "ideal allotment" or "quota." It is the number of seats that a state would ideally have, if fractional seats were possible. A state's quota is thus calculated by multiplying the size of the House (435) by the fraction that corresponds to the percentage of the U.S. population residing in that state.

The "apportionment problem" refers to the search for a method to replace these quotas by whole numbers in a way that is as fair and equitable as possible. Unfortunately, the naïve solution of just rounding each fraction to the nearest whole number fails because the resulting total will typically be either less than the fixed House size (leaving seats unfilled) or greater than the fixed House size (thus apportioning non-existent seats).

Alexander Hamilton, Secretary of the Treasury, proposed the first solution to the apportionment problem following the initial U.S. census in 1792. His proposal is easy to describe:

Hamilton's Method of Apportionment: Begin by rounding all quotas down to the nearest whole number and allocate seats accordingly, leaving (typically) a number of seats not yet allocated. Now hand these additional seats out, one at a time, according to the size of the fractional part of the quota (so that a state with a quota of 13.92 would get an extra seat before a state with a quota of 31.67, because .92 is greater than .67).

Hamilton's method was not used in 1792 because President George Washington vetoed the bill (the first bill in U.S. history to suffer this fate). It was, however, resurrected in 1850 and used for the next 40 years. In **Section 5.3**, we say more of the history of apportionment in the U.S., and we'll see that the choice of method to be used was often based more on political considerations than objective issues of fairness.

For the moment, let's ask what it might mean to say that a specific method, such as Hamilton's, for apportioning seats among the 50 states is "fair and equitable." Without some attempt to formalize this via desirable properties, we're back at the constitutional directive to do it "according to their respective numbers."

As a starting point, Hamilton's method possesses two properties that certainly seem, at first blush, to be obvious desiderata.

The Monotonicity Property

A method of apportionment satisfies the *monotonicity property* (or is said to be *monotone* or *monotonic*) if no state receives fewer seats than a state with less (or the same) population. That is, if state A has fewer seats than state B, then state A should have less population than state B.

The Quota Property

A method of apportionment satisfies the *quota property* (or, more briefly, satisfies *quota*) if the number of seats allotted to a state never differs from its (ideal) quota by more than one. Thus, if a state's quota is 13.92, it should receive either 13 seats or 14 seats.

Adding to the challenge of achieving fairness in apportioning seats is the fact that a census is conducted every 10 years, and so seats will typically have to be reallocated. Achieving fairness in view of such transitions turns out to be surprisingly difficult, and Hamilton's method comes up a bit short. In particular, it fails to satisfy the following (as we will later demonstrate).

The Population Property

A method of apportionment satisfies the *population property* (or *avoids the population paradox*) if, following a census, no state should gain population and lose a seat while some other state loses population and gains a seat.

It turns out that there is no shortage of apportionment methods that satisfy the population property. These are the so-called divisor methods.

5.3 DIVISOR METHODS OF APPORTIONMENT

In 1792, Hamilton's proposal was immediately met by a counter-proposal put forth by his chief political rival Thomas Jefferson. Jefferson's method (described below) seems to involve an enormous number of trial-and-error calculations, but this is not really true in practice. It is an example of a so-called "divisor method."

Jefferson's Method of Apportionment: Begin by choosing a whole number d (called a "divisor") as the desired size (population) of each "congressional district." Now allocate each state one seat for each congressional district. (That is, divide the state's population by the number d, and then round **down** to the nearest whole number to get that state's allocation.) If the number of seats allocated is exactly the House size, you're done. If it's less than the House size, go back and repeat the process with a larger choice for d. If it's more than the House size, use a smaller choice for d.

Politically, Jefferson's method won out, and it was used to apportion the House of Representatives for more than 50 years. However, by always rounding down, it systematically favors large states—a reduction from an ideal allotment of 49.9 to 49 leaves a state only about 2% short of ideal, whereas a reduction from an ideal allotment of 4.9 to 4 leaves a state about 20% short.

This favoritism of large states came to the forefront following the census of 1830 when John Quincy Adams, the Representative from Massachusetts, saw how badly the smaller New England states were faring in comparison with large states like New York. He proposed replacing Jefferson's method with one now known as Adams's method:

Adams's Method of Apportionment: Proceed exactly as in Jefferson's method except, where Jefferson rounds **down** to the nearest whole number, now round **up** to the nearest whole number.

Adams's method favors small states for exactly the same reason that Jefferson's method favors large states: a rounding up from 4.1 to 5 is a gain of roughly 20% while a rounding up from 49.1 to 50 is a gain of roughly only 2%. It was left to Daniel Webster to propose a more moderate alternative.

Webster's Method of Apportionment: Proceed exactly as in Adams's and Jefferson's methods except, where Jefferson rounds down and Adams rounds up, now simply round to the nearest whole number as one would normally do.

Historically, Webster's method went into effect following the 1840 census, but is was replaced a decade later by Hamilton's method, rediscovered by Samuel F. Vinton and often referred to as "Vinton's method." One might expect this to be the end of the story—at least for divisor methods—but there turns out to be one more natural way to round numbers, and the corresponding divisor method has been the one that has been in effect since the census of 1930. It uses the idea of the geometric mean.

DEFINITION. The *geometric mean* of two numbers A and B is the square root of the product AB. "Rounding according to the geometric mean" means, for example, that a number x between 4 and 5 gets rounded down to 4 if x is less than the geometric mean of 4 and 5 (i.e. $\sqrt{20}$), and rounded up otherwise. This rounds x down if $x < \sqrt{20}$ and up if $x > \sqrt{20}$.

The Hill-Huntington Method of Apportionment: Proceed exactly as in Adams's and Jefferson's and Webster's methods except, where Jefferson rounds down and Adams rounds up and Webster rounds in the normal fashion, now round according to the geometric mean.

The rationale behind the Hill-Huntington method can briefly be described as follows. In 1911, Joseph A. Hill, then the chief statistician in the census bureau, suggested a philosophical principle—as opposed to a method—on which apportionment should be based. He wanted to look at per capita representation, that is, a state's population divided by the number of seats. Thus, one state might have a per capita representation of 1,740,000 while another might have a per capita representation of 1,340,000 million. The difference, arrived at by subtracting, is 400,000. But what one really wants to look at here is the relative difference, in this case 400,000/1,340,000 = .2985 or 29.85%. It may happen that transferring a seat from the state with the smaller per capita representation to the one with the larger would reduce this relative difference, thus improving the equity. Hill's proposal was to find an apportionment method with the property that no two states could reduce the relative difference in per capita representation by such a transfer of a seat.

Edward V. Huntington, a professor of mathematics at Harvard, showed that the method now called the Hill-Huntington method does, in fact, satisfy Hill's principle.

5.4 A GLIMPSE OF IMPOSSIBILITY

There is a remarkable result due to Michel L. Balinski and H. Peyton Young that the only apportionment methods that satisfy the population property are the divisor methods. But divisor methods, it turns out, are never guaranteed to satisfy the quota condition. Thus, we have a situation—an impossibility theorem—analogous to what we saw in the context of social choice.

While the Balinski-Young result is quite complicated, there is a weaker result that is nevertheless striking. It shows that a search for a perfect apportionment method is as doomed as the search for a perfect social choice procedure.

THEOREM. *There is no apportionment method that satisfies the monotonicity property, the quota condition, and the population property.*

PROOF. Assume that we have an apportionment method that satisfies the monotonicity property and the quota condition. We'll show that it must fail to satisfy the population property.

Consider the situation in which there are 7 seats, 4 states (*A*, *B*, *C*, and *D*), and a total population of 4200 distributed as follows: *A* has 3003, *B* has 400, *C* has 399, and *D* has 398. We can calculate the quota for each in the usual way (for example, *A*'s quota is $[3003/4200] \times 7 = 5.005$). The results are as follows:

State	Population	Quota
A	3003	5.005
B	400	0.667
C	399	0.665
D	398	0.663

It is easy to see that because of the quota condition and monotonicity, the only possible apportionments are 5,1,1,0 and 6,1,0,0 (see Exercise 1). In particular, state *A* gets at least 5 seats and state *D* gets no seats.

Now suppose that at the next census there are 1100 additional people, with state *A* gaining 1, state *D* losing 1, and states *B* and *D* faring as show below:

State	Population	Quota
A	3004 (+1)	3.968
B	1503 (+1103)	1.985
C	396 (−3)	0.523
D	397 (−1)	0.524

Again, it is easy to see that because of the quota condition and monotonicity, the only possible apportionments are 4,2,0,1 and 4,1,1,1 and 3,2,1,1 (see Exercise 2). In particular, state *A* gets at most 4 seats and state *D* gets at least one.

Thus, state *A* has gained population and lost a seat, while state *D* has lost population and gained a seat. This completes the proof.

5.5 DISPUTE RESOLUTION AND FAIR DIVISION

In Chapter 1, we studied social choice procedures, and evaluated each according to a set of reasonably fair criteria that we intuitively believe a social choice procedure should satisfy. In the last few sections, we looked at apportionment methods and again evaluated each according to a set of reasonably fair criteria that we intuitively believe an apportionment method should satisfy. Next we look at the issue of fairness in a different realm—dispute resolution. We will consider several methods of dispute resolution, and again evaluate each according to different notions of fairness.

Even most children are familiar with the method of "divide-and-choose." If two people want to fairly divide a candy bar in two pieces, one person will physically divide the candy, and the second person will choose which piece to take. Since the divider doesn't know which piece he will receive, it is clearly in his best interest to make the two pieces equal size, thereby guaranteeing that he will receive half. The chooser is also happy, of course, since she will definitely get the bigger piece. This method of division works equally well if the item to be divided is heterogeneous and the people's valuations of each piece differ. For example, we may be dividing a birthday cake into two pieces—one of us just wants as big a piece as possible, while the other would rather have a smaller piece if it contains more of the delicious frosting roses. The divider might split the cake into two pieces of unequal size, but if he values each piece equally, he will still be happy in the end since he is guaranteed a piece worth half the total value.

The divide-and-choose method described above is well known and commonly used. In fact, it has been used for thousands of years! In the ancient Greek text *Theogeny* written by Hesiod over 2700 years ago, Prometheus and Zeus divide some meat between them; Prometheus