MA 665 EXERCISES 7

(1) Let $\beta_{i,j}$ be the graded Betti numbers of a finitely generated S-module. If, for a given i there is a d such that $\beta_{i,j} = 0$ for all $j < d$, show that then $\beta_{i+1,j+1} = 0$ for all $j < d$.

(2) Let $\beta_{i,j}$ be the graded Betti numbers of a finitely generated S-module, and let $B_j = \sum_{i \geq 0} (-1)^i \beta_{i,j}$. Show that

$$H_M(d) = \sum_j B_j \binom{r+d-j}{r}.$$

Moreover, the values B_j can be determined inductively from the Hilbert function by the formula

$$B_j = H_M(j) - \sum_{k<j} B_k \binom{r+j-k}{r}.$$

(3) Consider the map from $S = k[x_0, x_1, x_2, x_3]$ to $k[y, z]$ given by sending x_i to $y^i z^{3-i}$ for all i, and let T denote the image. The following series of exercises shows that a resolution of T is given by

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to T \to 0.$$

(a) Show that $H_T(d) = 3d + 1$ for $d \geq 0$.
(b) Compute the Hilbert functions of the terms S, $S(-2)^3$, and $S(-3)^2$. Show that their alternating sum is equal to the Hilbert function H_T.
(c) Show that the map from $S(-3)^2$ to $S(-2)^3$ given by the matrix

$$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$

is injective.
(d) Show that the results in parts (b) and (c) together imply that the complex above is a free resolution of T.