

SLIDING BLOCK PUZZLES WITH A TWIST: ON SEGERMAN'S 15+4 PUZZLE

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ABSTRACT. Segerman's 15+4 puzzle is a hinged version of the classic 15-puzzle, in which the tiles rotate as they slide around. In 1974, Wilson classified the groups of solutions to sliding block puzzles. We generalize Wilson's result to puzzles like the 15+4 puzzle, where the tiles can rotate, and the sets of solutions are subgroups of the generalized symmetric groups. Aside from two exceptional cases, we see that the group of solutions to such a puzzle is always either the entire generalized symmetric group or one of two special subgroups of index two.

1. INTRODUCTION

1.1. Sliding Block Puzzles. The classic 15-puzzle, pictured in Figure 1, is a 4×4 grid containing 15 tiles, labeled with the numbers 1 through 15, and one empty square. You can move the tiles around by sliding an adjacent tile into the empty square, and the goal is to arrange the tiles in order from 1 to 15 via a sequence of these moves. A natural question is: given a starting permutation of the tiles, is it always possible to solve the puzzle? If not, which permutations are solvable? In 1879, Johnson and Story showed that the 15-puzzle cannot be solved if the starting permutation is odd [4]. Thus, for example, if you were to pop out two of the tiles, switch their places, and pop them back in again, the resulting puzzle would be impossible to solve.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIGURE 1. The 15-puzzle

In 1974, Wilson considered more general sliding block puzzles on arbitrary graphs, where the vertices represent the possible positions of the tiles and there is an edge between two vertices if the corresponding tiles are adjacent [7]. If the graph has $n + 1$ vertices, then the set of solvable permutations in which the empty vertex is fixed forms a subgroup of the symmetric group S_n . Wilson's main result is a classification of these subgroups.

Theorem 1. [7, Theorem 2] *Let Γ be a simple, 2-vertex connected graph on $n + 1$ vertices that is not a cycle. Then the group of solvable permutations is:*

- A_n if Γ is bipartite,

- $\mathrm{PGL}(2, 5)$ if Γ is the graph Θ_7 pictured in Figure 2, and
- S_n otherwise.

For example, the graph corresponding to the 15-puzzle is bipartite, so the set of solvable permutations is the alternating group A_{15} . Here, $\mathrm{PGL}(2, 5) \cong S_5$ is the “exotic” subgroup of S_6 isomorphic to S_5 . It acts on the vertices of Θ_7 , labeled as in Figure 2, by fractional linear transformations.

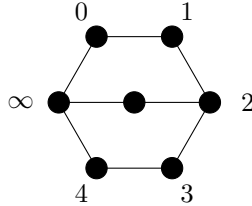


FIGURE 2. The graph Θ_7

Remark. The assumption that Γ is simple in Theorem 1 is not necessary. If a pair of vertices in Γ have multiple edges between them, then sliding a tile along any of these edges results in the same permutation of the tiles. Thus, if Γ has multi-edges, the group of solvable permutations is the same as that of the simple graph obtained by replacing each multi-edge in Γ by a single edge.

1.2. **The 15 + 4 Puzzle.** In 2021, Segerman created the 15 + 4 puzzle, pictured in Figure 3. Much like the 15-puzzle, Segerman’s puzzle consists of one empty vertex and 19 tiles, which can be moved by sliding an adjacent tile into the empty spot. The twist is that moving the tiles around can result in some of them becoming rotated. For example, in Figure 4, the tiles labeled 6, 7, 10, and 16 are rotated 90 degrees from their starting positions. The goal is not only to get the tiles in the correct order, but with the correct rotations as well. More details on the 15 + 4 puzzle are provided in Dr. Segerman’s YouTube video:

<https://www.youtube.com/watch?v=Hc3yfuXiWe0>.

The goal of this note is to generalize Wilson’s result to sliding block puzzles in which the tiles rotate as they slide around. We assume throughout that each tile has m sides, so it can rotate m times. Each position of the puzzle determines an ordered pair (\vec{x}, σ) , where $\sigma \in S_n$ is a permutation and $\vec{x} \in (\mathbb{Z}/m\mathbb{Z})^n$ records the rotations of the tiles. As we explain in greater detail in Section 3, the set of such ordered pairs forms a subgroup of the generalized symmetric group $S(m, n)$.

Figure 5 depicts a graphical representation of Segerman’s 15+4 puzzle. Here, each tile has 4 sides. If a tile slides in either direction across one of the solid lines, then it does not rotate. If it slides across one of the dashed lines from left to right, it rotates 90 degrees, and if it slides from right to left, it rotates 90 degrees in the other direction. Because the graph is not bipartite, by Theorem 1, it is possible to obtain every permutation of the tiles from any starting position. As we will see in Theorem 2 below, however, it is not possible to obtain every possible ordered pair of permutations and rotations. In particular, in any odd permutation of the tiles, some of the tiles will be non-trivially rotated.

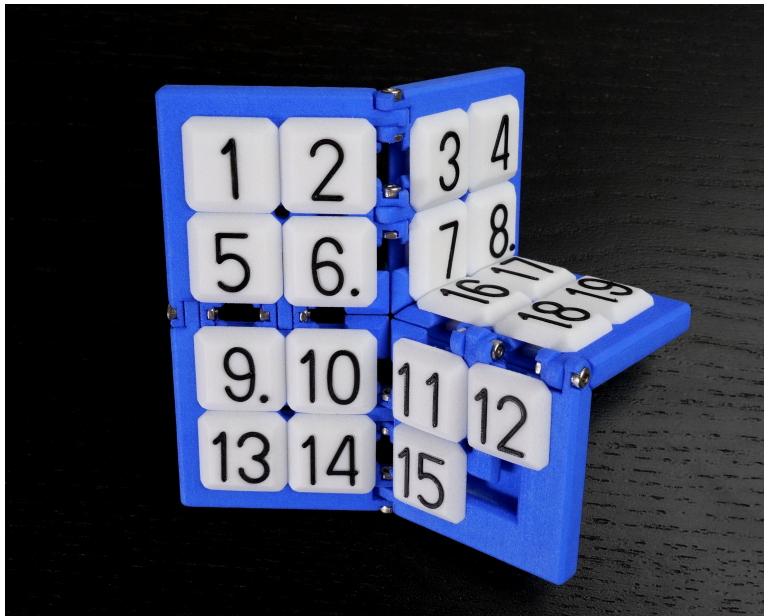


FIGURE 3. Segerman's 15 + 4 Puzzle (Photo courtesy of Henry Segerman)

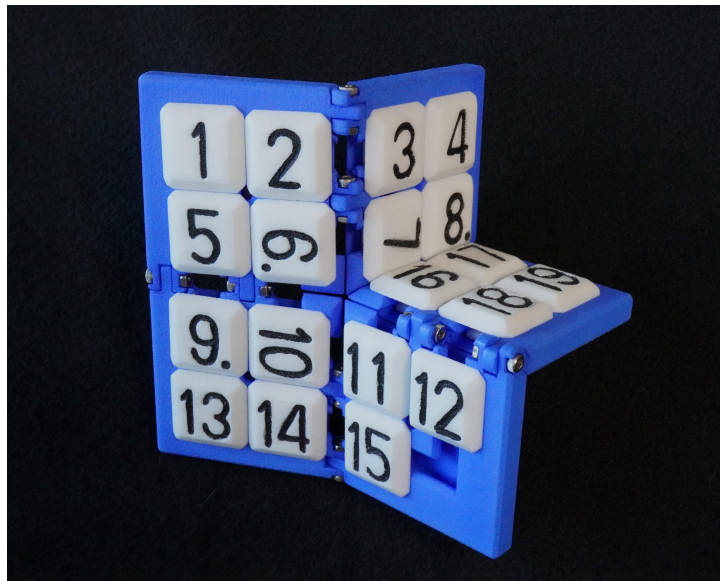
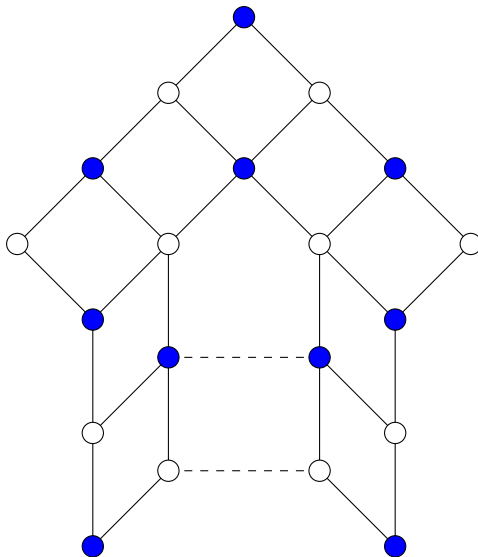


FIGURE 4. A solvable state of the 15 + 4 puzzle, in which some of the tiles are rotated

1.3. **Twist Graphs.** Following this example, to keep track of how a puzzle rotates the tiles, we decorate each edge of the graph with an element of $\mathbb{Z}/m\mathbb{Z}$, which records how much a tile rotates when it slides along that edge. More precisely,

FIGURE 5. Twist graph representation of the $15 + 4$ puzzle

recall that a *1-chain* γ on a graph Γ with coefficients in $\mathbb{Z}/m\mathbb{Z}$ consists of an element $\gamma_e \in \mathbb{Z}/m\mathbb{Z}$ for each oriented edge e in Γ , subject to the condition that, if e and \bar{e} represent the same edge with opposite orientations, then $\gamma_{\bar{e}} = -\gamma_e$. We define a *twist graph* to be an ordered pair (Γ, γ) , where Γ is a graph and γ is a 1-chain on Γ with coefficients in $\mathbb{Z}/m\mathbb{Z}$. Returning to our example, Figure 5 represents the $15 + 4$ puzzle by a twist graph (Γ, γ) . Here, $m = 4$, the solid lines represent edges $e \in E(\Gamma)$ with $\gamma_e \equiv 0 \pmod{4}$, and the dashed lines represent oriented edges e with $\gamma_e \equiv 1 \pmod{4}$ when oriented from left to right.

We say that a twist graph (Γ, γ) is *twist bipartite* if m is even and the vertices of Γ can be colored using two colors such that:

- (1) if γ_e is even, then the endpoints of e have different colors, and
- (2) if γ_e is odd, then the endpoints of e have the same color.

Equivalently, a twist graph is twist bipartite if the number of even edges in any cycle is even. For example, the coloring of the vertices in Figure 5 illustrates that the pictured twist graph is twist bipartite. In particular, if one deletes the two dashed edges, then the resulting graph is bipartite, and in any 2-coloring of the vertices of this subgraph, the dashed edges connect vertices of the same color.

Our main result requires an assumption on the homology of the graph, which we now briefly explain. The significance of this condition will be explored in greater detail at the end of Section 2. A 1-chain ω on a graph Γ is a *1-cycle* if, for all vertices $v \in V(\Gamma)$, we have $\sum_{e \rightarrow v} \omega_e = 0$, where the sum is over all oriented edges with head v [3, p. 105]. The group of 1-cycles on Γ with coefficients in $\mathbb{Z}/m\mathbb{Z}$ is called the *first homology group* of Γ , and denoted $H_1(\Gamma, \mathbb{Z}/m\mathbb{Z})$. Given a twist graph (Γ, γ) , there is a natural homomorphism $\varphi_\gamma: H_1(\Gamma, \mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{Z}/m\mathbb{Z}$ given

by

$$\varphi_\gamma(\omega) = \sum_{e \in E(\Gamma)} \gamma_e \cdot \omega_e \pmod{m}.$$

Throughout, we will restrict our attention to twist graphs for which the homomorphism φ_γ is surjective. In Lemmas 2 and 3 of the next section, and the remarks immediately thereafter, we will see that we can always reduce to this case. Note that if m is even and φ_γ is surjective, then there is at least one cycle in Γ with an odd number of odd edges. Thus, the twist graph (Γ, γ) cannot be both bipartite and twist bipartite.

Our main result is a classification of the possible permutations and rotations that can occur in sliding block puzzles where the tiles can rotate.

Theorem 2. *Let (Γ, γ) be a 2-vertex connected twist graph with no loops and with φ_γ surjective. Suppose that Γ is not a cycle, the graph Θ_5 of Figure 6, the graph Θ_7 of Figure 2, or any graph obtained from one of these three by replacing edges with multiple parallel edges. Then the group of solvable ordered pairs $(\vec{x}, \sigma) \in S(m, n)$ is:*

- $\{(\vec{x}, \sigma) \in S(m, n) \mid \sigma \in A_n\}$ if Γ is bipartite,
- $\{(\vec{x}, \sigma) \in S(m, n) \mid \sum_{i=1}^n x_i \equiv \text{sign}(\sigma) \pmod{2}\}$ if m is even and (Γ, γ) is twist bipartite, and
- $S(m, n)$ otherwise.

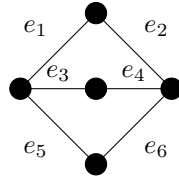


FIGURE 6. The graph Θ_5

If either Γ is bipartite or (Γ, γ) is twist bipartite, then exactly half of the ordered pairs $(\vec{x}, \sigma) \in S(m, n)$ are solvable, but for different reasons. For example, the twist graph that represents the 15+4 puzzle, pictured in Figure 5, is twist bipartite. Thus, by Theorem 2, the set of solvable ordered pairs $(\vec{x}, \sigma) \in S(4, 19)$ for the 15 + 4 puzzle is

$$\left\{ (\vec{x}, \sigma) \in S(4, 19) \mid \sum_{i=1}^{19} x_i \equiv \text{sign}(\sigma) \pmod{2} \right\}.$$

Thus, if you were to pop out one of the tiles, rotate it 90 degrees, and pop it back in again, the resulting puzzle would be unsolvable. Similarly, if you were to pop out two of the tiles and swap them without rotating, the resulting puzzle would be unsolvable. If, however, you were to pop out two of the tiles, swap them, and rotate one of them 90 degrees before popping them back in, the resulting puzzle would be solvable.

1.4. Exceptional Cases. If Γ is a cycle, then the group of solvable permutations is cyclic. In addition to the cycle graphs, Theorem 2 has two exceptional cases. Both of these are related to well-known exotic behaviors of the small-order symmetric and alternating groups. Famously, A_4 is the only alternating group that is not simple. The Klein 4-group K is a normal subgroup of A_4 , and we denote the quotient by

$$q: A_4 \rightarrow A_4/K \cong \mathbb{Z}/3\mathbb{Z}.$$

Additionally, S_6 is the only symmetric group with a non-trivial outer automorphism. This automorphism maps the “standard” copy of S_5 in S_6 isomorphically onto the “exotic” copy $\text{PGL}(2, 5) \cong S_5$. The following characterizes the possible permutations and rotations that can occur in sliding block puzzles where the underlying graph is one of the two exceptional graphs Θ_5 or Θ_7 .

Theorem 3. *If $\Gamma = \Theta_5$ and φ_γ is surjective, then the group of solvable order pairs $(\vec{x}, \sigma) \in S(m, 4)$ is:*

- (1) $\left\{ (\vec{x}, \sigma) \in S(m, 4) \mid \sigma \in A_4, \sum_{i=1}^4 x_i \equiv q(\sigma) \pmod{3} \right\}$
if m is divisible by 3 and $\sum_{i=1}^6 \gamma_{e_i} \equiv 0 \pmod{3}$, and
- (2) $\left\{ (\vec{x}, \sigma) \in S(m, 4) \mid \sigma \in A_4 \right\}$ otherwise.

If $\Gamma = \Theta_7$ and φ_γ is surjective, then the group of solvable order pairs $(\vec{x}, \sigma) \in S(m, 6)$ is:

- (1) $\left\{ (\vec{x}, \sigma) \in S(m, 6) \mid \sigma \in \text{PGL}(2, 5), \sum_{i=1}^6 x_i \equiv \text{sign}(\sigma) \pmod{2} \right\}$
if m is even and (Θ_7, γ) is twist bipartite, and
- (2) $\text{PGL}(2, 5)$ otherwise.

1.5. Other Puzzles. The 15+4 puzzle is not the only puzzle of this type. Segerman’s other creations include the hyperbolic 29-puzzle and the continental drift puzzle, pictured in Figures 7 and 8, which he discusses in the YouTube videos:

<https://www.youtube.com/watch?v=EitWHthBY30>

<https://www.youtube.com/watch?v=0uQx33KFM00>.

The latter puzzle is a globe that is tiled like a soccer ball, where the 12 hexagonal faces are tiles that can slide around. As in the 15 + 4 puzzle, these are sliding block puzzles in which sliding the tiles may cause them to rotate. Online versions of these and similar puzzles, written by the makers of HyperRogue, can be found at

<http://roguetemple.com/z/15/>.

Remark. Given a puzzle like the 15 + 4 puzzle or the continental drift puzzle, how do we construct the corresponding twist graph? To start, we need to record how much a tile has been rotated even when that tile is not in its original position. To do this, for each tile we choose one of its m sides to call the “top” of the tile. No matter where a tile is, it is unrotated if its top side lines up with the top side of the tile that is there in the puzzle’s solved state. In the 15 + 4 puzzle, where the tiles have numbers written on them, there is a natural choice of top side for each tile. If we consider the continental drift puzzle, however, it becomes clearer that this is an arbitrary choice. For example, no side of the North Pole tile has a particularly strong claim to being its “top”. If we slide the tiles around so that the Madagascar tile is where the North Pole should be, none of the 6 possible rotations



FIGURE 7. Segerman's Hyperbolic 29-Puzzle (Photo courtesy of Henry Segerman)



FIGURE 8. Segerman's Continental Drift Puzzle (Photo courtesy of Henry Segerman)

of the Madagascar tile is the natural “correct” one. Thus, designating a top side for each tile forces us to make a choice. The corresponding twist graph depends on how one makes this choice.

The graphs representing the hyperbolic 29-puzzle and the continental drift puzzle are pictured in Figure 9. In the first case, $m = 4$ and in the second, $m = 6$. Both graphs contain a simple closed path of length 5, so neither is bipartite. Following the remark above, one can choose the “top” side of each tile so that $\gamma_e \equiv \pm 1 \pmod{m}$ for every oriented edge e in Γ . Specifically, in Figure 9, the edges are labeled with arrows so that $\gamma_e \equiv 1 \pmod{m}$ when e is oriented in the direction of the arrow. Since γ_e is odd for all e , both twist graphs are twist bipartite. Finally, both graphs are planar, and if $\omega \in H_1(\Gamma, \mathbb{Z}/m\mathbb{Z})$ is a face of the planar embedding, then $\varphi_\gamma(\omega) \equiv \pm 1 \pmod{m}$. (When $m = 4$, this follows immediately from the fact that $\gamma_e \equiv \pm 1 \pmod{m}$ for all oriented edges e . When $m = 6$, it suffices to check that no pentagonal face has exactly 4 edges oriented either clockwise or counter-clockwise.) Therefore, φ_γ is surjective. Thus, by Theorem 2, the set of solvable order pairs for the hyperbolic 29-puzzle is

$$\left\{ (\vec{x}, \sigma) \in S(4, 29) \mid \sum_{i=1}^{29} x_i \equiv \text{sign}(\sigma) \pmod{2} \right\},$$

and the set of solvable ordered pairs for the continental drift puzzle is

$$\left\{ (\vec{x}, \sigma) \in S(6, 11) \mid \sum_{i=1}^{11} x_i \equiv \text{sign}(\sigma) \pmod{2} \right\}.$$

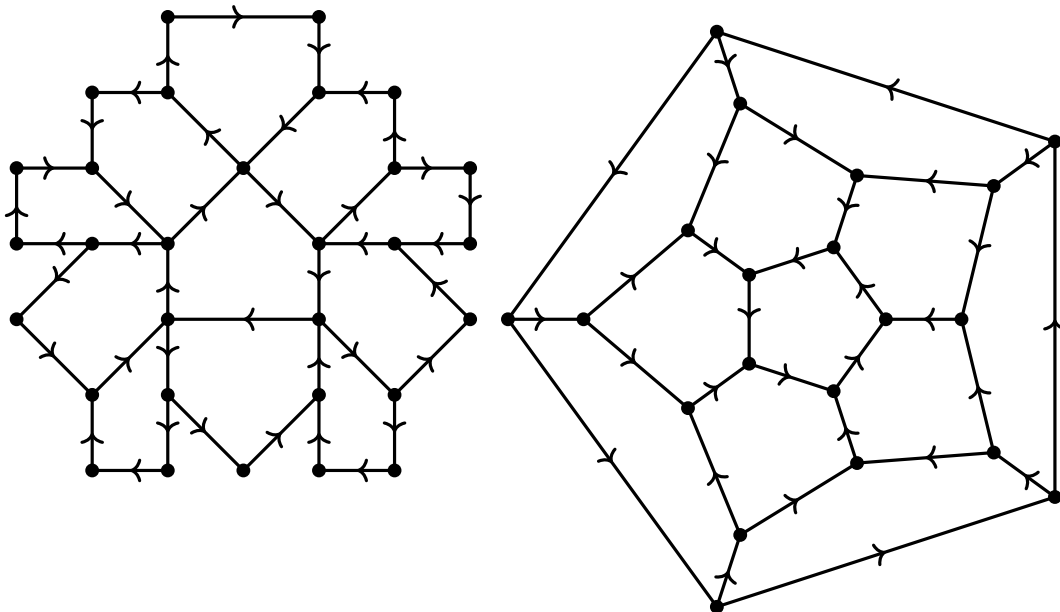


FIGURE 9. Graph representations of the hyperbolic 29-puzzle (left) and the continental drift puzzle (right)

In the puzzles we study here, each tile has a cyclic group of rotations, but one can imagine other possibilities. Figure 10 depicts a sliding block puzzle whose “tiles” are cubes. The group of rotations of each individual tile is the rotation group of the

cube, which is isomorphic to S_4 . One potential line of future inquiry could be to study puzzles of this type, where the group of rotations of the tiles are not cyclic.

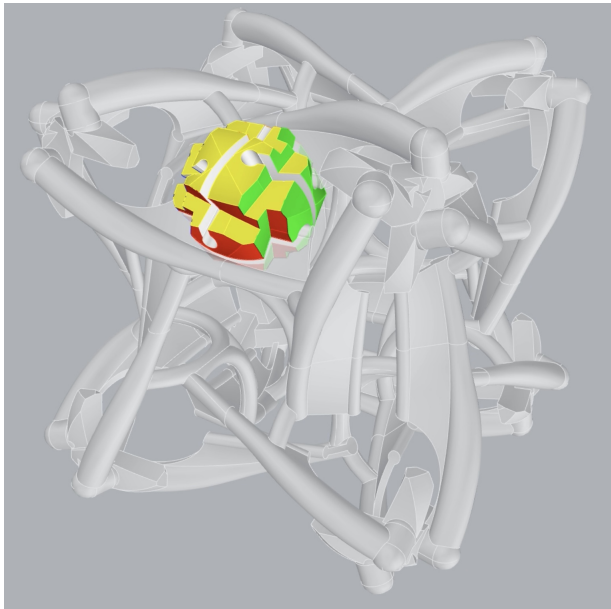


FIGURE 10. A sliding block puzzle whose tiles are cubes (Photo courtesy of Henry Segerman)

1.6. Outline of the paper. In Section 2, we describe Wilson's construction of the group of solvable permutations, and some basic properties of the fundamental groups of graphs. In Section 3, we discuss the generalized symmetric groups $S(m, n)$, and see that the set of solvable order pairs is a subgroup of $S(m, n)$. Finally, in Section 4, we prove Theorems 2 and 3.

2. THE FUNDAMENTAL GROUP

In this section, we briefly describe the fundamental group of a graph and Wilson's construction of a homomorphism from this group to the symmetric group. The fundamental group can be defined for more general (pointed) topological spaces (see, for example [3, Proposition 1.3]), but for our purposes, we will only need the fundamental groups of graphs.

2.1. Paths and Homotopy. A *path* p in a graph Γ is a sequence of oriented edges

$$p = e_1 e_2 \cdots e_k$$

such that the head of e_i is equal to the tail of e_{i+1} for all i . The path p is called a *closed path* if the head of e_k is equal to the tail of e_1 . In other words, a closed path is a path that starts and ends at the same vertex. The path p is called *simple* if the heads of the edges e_i are distinct and the tails of e_i are distinct. That is, a path is simple if it does not pass through a vertex more than once, with the possible exception that it may start and end at the same vertex.

If two closed paths $p = e_1 \dots e_k, q = f_1 \dots f_\ell$ both start and end at the same vertex v , then their concatenation $pq = e_1 \dots e_k f_1 \dots f_\ell$ is a closed path that starts and ends at v . This defines a product on the set of closed paths that start and end at the given vertex v . This does not, however, define a group structure on this set, since there are no inverses. To fix this, we define an equivalence relation on this set.

The closed path $p = e\bar{e}$ consisting of an oriented edge, followed by that same edge with the opposite orientation, is called the *irrelevant closed path*. We say that two paths are *homotopy equivalent* if one can be obtained from the other by a sequence of inserting and deleting irrelevant closed paths. The *fundamental group* $\pi_1(\Gamma, v)$ is the set of homotopy equivalence classes of closed paths that start and end at the vertex v , under the operation of concatenation. The identity element of this group is the empty path, and the inverse of a path $p = e_1 \dots e_k$ is the *reverse path* $\bar{p} = \bar{e}_k \dots \bar{e}_1$.

The first homology group $H_1(\Gamma, \mathbb{Z})$ defined in Section 1 is the abelianization of $\pi_1(\Gamma, v)$. In particular, any homomorphism from $\pi_1(\Gamma, v)$ to an abelian group factors through $H_1(\Gamma, \mathbb{Z})$. Moreover, any homomorphism from $\pi_1(\Gamma, v)$ to an m -torsion abelian group factors through $H_1(\Gamma, \mathbb{Z}/m\mathbb{Z})$.

2.2. Wilson's Construction. Given an oriented edge e in a graph Γ , Wilson defines σ_e to be the permutation of $V(\Gamma)$ that transposes the head and tail of e . For a path $p = e_1 \dots e_k$, he then defines the permutation σ_p by composing¹ the transpositions

$$\sigma_p = \sigma_{e_k} \circ \dots \circ \sigma_{e_2} \circ \sigma_{e_1}.$$

If Γ represents a sliding block puzzle and the path p starts at the empty vertex, then σ_p is the permutation of the tiles obtained by sliding the tiles along the path p . In other words, a tile that is initially located at a vertex w will move to $\sigma_p(w)$ after performing this sequence of moves. Note that, if p is a simple closed path of length k , then σ_p is the $(k-1)$ -cycle

$$\sigma_p = \left(h(e_1)h(e_2) \dots h(e_{k-1}) \right),$$

where $h(e_i)$ denotes the head of e_i .

Because every transposition is its own inverse, the map σ takes homotopy equivalent paths to the same permutation. Thus, it descends to a well-defined map on the fundamental group $\pi_1(\Gamma, v)$. By construction, this map is a homomorphism, and every permutation in the image fixes the vertex v . The group of solvable permutations in which the empty vertex is fixed is simply the image of the homomorphism σ . The possibilities for this image are classified in Theorem 1.

Remark. The set of solvable permutations that do *not* fix the empty vertex does not form a group, because one can only concatenate two permutations if the position of the empty vertex at the end of the first permutation agrees with that at the start of the second permutation. This set does form an arguably more natural object called a *groupoid*, but this is beyond the scope of the present article [6, Definition 1.1.11].

¹In his paper, Wilson uses the convention that permutations are composed from left to right, whereas we prefer to compose them from right to left.

2.3. Generators and the Map φ_γ . The fundamental group of a graph has a well-known set of generators.

Lemma 1. *Let Γ be a connected graph, and let v be any vertex in Γ . Then $\pi_1(\Gamma, v)$ is freely generated by g simple closed paths, where $g = |E(\Gamma)| - |V(\Gamma)| + 1$. Moreover if Γ is not a cycle, or obtained from a cycle by replacing an edge with multiple parallel edges, then these simple closed paths can be chosen so they are non-Hamiltonian.*

Proof. The first part is standard. Let $T \subseteq \Gamma$ be a spanning tree, and let e_1, \dots, e_g be the edges of Γ that are not contained in T . For each e_i , there is a unique simple path in T from the head of e_i to its tail. Appending e_i to the end of this path, we obtain a simple closed path p_i in Γ . By [3, Proposition 1A.2], the fundamental group $\pi_1(\Gamma)$ is the free group on the generators p_1, \dots, p_g .

For the second part, if Γ is not a cycle, or obtained from a cycle by replacing an edge with multiple parallel edges, then either Γ is a tree, in which case the fundamental group is trivial, or there exists a vertex w in Γ that has at least 3 neighbors. Construct a spanning tree T by breadth-first search starting from w . By construction, the simple closed paths p_i contain at most 2 of the neighbors of w . Hence, they are non-Hamiltonian. \square

Lemma 1 allows us to describe a large family of twist graphs (Γ, γ) for which φ_γ is surjective. Specifically, consider the subgraph $\Gamma_0 \subseteq \Gamma$ with $V(\Gamma_0) = V(\Gamma)$ and $E(\Gamma_0) = \{e \in E(\Gamma) \mid \gamma_e \equiv 0 \pmod{m}\}$. In other words, Γ_0 is the subgraph of Γ consisting of edges that do not rotate the tiles. We first note the following.

Lemma 2. *Every sliding block puzzle can be represented by a twist graph (Γ, γ) such that Γ_0 is connected.*

Proof. The graph Γ is uniquely determined by the puzzle – it has a vertex for every possible position of the tiles and an edge between two vertices if the corresponding vertices are adjacent. Let $T \subseteq \Gamma$ be a rooted spanning tree. Now, arbitrarily choose a side of the tile at the root vertex to be the “top” of the tile. We then inductively choose the top side of each tile so that $\gamma_e = 0$ for all edges e in T . Specifically, let v be a vertex for which we have already chosen a top side, let w be connected to v by an edge e in T , and suppose that we have not already chosen a top side for w . We can then choose the top side of the tile at w so that $\gamma_e = 0$. Proceeding in this way from the root vertex to each vertex in Γ , we see that $T \subseteq \Gamma_0$, so Γ_0 is connected. \square

Lemma 2 is important for the following reason.

Lemma 3. *Suppose that Γ_0 is connected. The map φ_γ is surjective if and only if the set $\{\gamma_e \mid e \in E(\Gamma)\}$ generates $\mathbb{Z}/m\mathbb{Z}$. Equivalently, φ_γ is surjective if and only if the set $\{m, \gamma_e \mid e \in E(\Gamma)\}$ has no common divisor greater than 1.*

Proof. Since Γ_0 is connected, it contains a spanning tree T . The subgraph Γ_0 has the same set of vertices as Γ , so T is also a spanning tree for Γ . In the construction of Lemma 1, each of the simple closed paths p_i contains exactly 1 edge that is not in the spanning tree. Thus, each of the simple closed paths p_i contains at most 1 edge e_i such that $\gamma_{e_i} \not\equiv 0 \pmod{m}$. Because these simple closed paths generate $\pi_1(\Gamma, v)$, their images in first homology generate $H_1(\Gamma, \mathbb{Z})$. It follows that the map φ_γ is surjective if and only if the elements $\gamma_{e_1}, \dots, \gamma_{e_g}$ generate $\mathbb{Z}/m\mathbb{Z}$. The last

statement holds because the subgroups of $\mathbb{Z}/m\mathbb{Z}$ are precisely the cyclic subgroups generated by divisors of m . \square

For example, consider the twist graph (Γ, γ) pictured in Figure 5, representing the $15 + 4$ puzzle. The subgraph Γ_0 is obtained by deleting the two dashed edges. We see that Γ_0 is connected. If e is one of the dashed edges, oriented from left to right, then $\gamma_e \equiv 1 \pmod{4}$, which generates $\mathbb{Z}/4\mathbb{Z}$. Thus, φ_γ is surjective.

On the other hand, if there exists an integer $a > 1$ dividing both m and γ_e for all $e \in E(\Gamma)$, then the only possible rotations are multiples of a . It is therefore natural to recast the problem – instead of the tiles having m sides, one should think of them as having $\frac{m}{a}$ sides, and we obtain a twist graph with coefficients in $\mathbb{Z}/\frac{m}{a}\mathbb{Z}$ by dividing γ_e by a for all e . Reducing the problem in this way, one can apply Theorem 2 to all twist graphs for which Γ_0 is connected.

3. GENERALIZED SYMMETRIC GROUPS

3.1. Examining the group structure. In this section, we generalize Wilson’s construction to record both the permutation of the tiles and their possible rotations. Given a path p in the graph Γ , we will define an ordered pair $(\vec{x}(p), \sigma_p)$, where σ_p is the permutation of the previous section, and $\vec{x}(p)$ is a vector with coefficients in $\mathbb{Z}/m\mathbb{Z}$. First, however, we need to define the group structure on this set of ordered pairs. Importantly, we will see that the product group $(\mathbb{Z}/m\mathbb{Z})^n \times S_n$ is not appropriate for the task, because in sliding block puzzles the permutations and rotations do not commute.

To see this, consider the sliding block puzzle pictured on the left in Figure 11. In this example, $m = 2$, and sliding a tile across the dashed edge rotates that tile by 180 degrees. The empty vertex is the upper left one. The closed path p starts at the empty vertex, proceeds to the right along the solid edge, and then back to the left along the dashed edge. Sliding the tiles along the closed path p does not permute the tiles, but does rotate the tile labeled 1. The closed path q starts at the empty vertex and proceeds counter-clockwise around the solid triangle. Sliding the tiles along the closed path q transposes the tiles labeled 1 and 2, but does not rotate any tiles. On the righthand side of the figure, we see how the tiles are rotated and permuted by the closed paths pq (on the top) and qp (on the bottom). Crucially, we see that they are not the same.

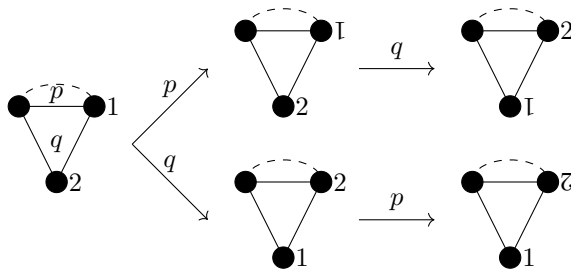


FIGURE 11. Permutations and rotations do not commute

To understand the group structure on this set of ordered pairs, first note that the map σ of the previous section should factor through this group. Thus, the kernel

$(\mathbb{Z}/m\mathbb{Z})^n$, consisting of ordered pairs of the form (\vec{x}, id) , is a normal subgroup. By [2, Theorem 5.12], the group is a semidirect product $(\mathbb{Z}/m\mathbb{Z})^n \rtimes S_n$. To determine the full group structure, we need to consider the action by conjugation of S_n on this normal subgroup. To that end, and generalizing the previous example, suppose we have two closed paths p and q in an arbitrary sliding block puzzle. Suppose further that sliding the tiles along p rotates some of the tiles but does not permute them, and conversely, sliding tiles along q permutes the tiles but does not rotate them. Consider what happens when we slide the tiles along the composite path $qp\bar{q}$. We see that the tile located at a vertex v first slides to vertex $\sigma_q(v)$, where it is then rotated by $\vec{x}(p)_{\sigma_q(v)}$, and then moved back to the vertex v . Thus,

$$\vec{x}(qp\bar{q})_v = \sigma_q^{-1} \cdot \vec{x}(p) \cdot \sigma_q = \vec{x}(p)_{\sigma_q(v)}.$$

In other words, S_n acts on $(\mathbb{Z}/m\mathbb{Z})^n$ by reindexing.

3.2. Generalized symmetric groups. This action completely determines the group structure on our semidirect product – it is the wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr S_n$, commonly known as the *generalized symmetric group* $S(m, n)$. The generalized symmetric groups have been studied extensively, dating back at least to Coxeter in 1936 [1, Section 5]. The generalized symmetric group $S(m, n)$ is isomorphic to the group of generalized permutation matrices (that is, matrices in which each row and each column has exactly one nonzero entry), where the nonzero entries are m th roots of unity.

The generalized symmetric group has several natural quotients, which we record here. First, it surjects onto the symmetric group:

$$\pi : S(m, n) \rightarrow S_n \quad \pi(\vec{x}, \sigma) = \sigma.$$

For any integer a dividing m , we have the maps:

$$\begin{aligned} \rho_{m,a} : S(m, n) &\rightarrow S(a, n) & \rho_{m,a}(\vec{x}, \sigma) &= (\vec{x} \pmod{a}, \sigma), \\ \eta_{m,a} : S(m, n) &\rightarrow \mathbb{Z}/a\mathbb{Z} & \eta_{m,a}(\vec{x}, \sigma) &= \sum_{i=1}^n x_i \pmod{a}. \end{aligned}$$

These maps fit into the following commutative diagram.

$$\begin{array}{ccccc} S(m, n) & & & & \\ \downarrow \eta_{m,m} & \searrow \rho_{m,a} & & \searrow \pi & \\ \mathbb{Z}/m\mathbb{Z} & & S(a, n) & \xrightarrow{\pi} & S_n \\ & \searrow \eta_{m,a} & \downarrow \eta_{a,a} & & \\ & & \mathbb{Z}/a\mathbb{Z} & & \end{array}$$

In the special case where $m = 2$, the generalized symmetric group $S(2, n)$ is also known as the *signed permutation group* or the *hyperoctahedral group*, and is isomorphic to the Coxeter group of type $B_n = C_n$. The subgroup appearing in the introduction,

$$\left\{ (\vec{x}, \sigma) \in S(2, n) \mid \eta_{2,2}(\vec{x}, \sigma) \equiv \text{sign}(\sigma) \pmod{2} \right\},$$

is isomorphic to the Coxeter group of type D_n .

3.3. Generalizing Wilson's construction. Returning to our construction, given an oriented edge e in a twist graph (Γ, γ) , we define

$$\vec{x}(e) \in \prod_{v \in V(\Gamma)} \mathbb{Z}/m\mathbb{Z}$$

to be the vector

$$\vec{x}(e)_v = \begin{cases} \gamma_e & \text{if } v \text{ is the tail of } e, \\ 0 & \text{otherwise.} \end{cases}$$

As in Wilson's construction, for a path $p = e_1 \cdots e_k$, we define the ordered pair $(\vec{x}(p), \sigma_p)$ by multiplying the ordered pairs

$$(\vec{x}(p), \sigma_p) := (\vec{x}(e_k), \sigma_{e_k}) \cdots (\vec{x}(e_2), \sigma_{e_2}) \cdot (\vec{x}(e_1), \sigma_{e_1})$$

in the generalized symmetric group $S(m, n)$. If (Γ, γ) represents a sliding block puzzle and the path p starts at the empty vertex, then $\vec{x}(p)$ records the rotations of the tiles obtained by sliding the tiles along the path p .

Because $\mathbb{Z}/m\mathbb{Z}$ is an m -torsion abelian group, the composition of the map from $\pi_1(\Gamma, v)$ to $S(m, n)$ with the map $\eta_{m,m} : S(m, n) \rightarrow \mathbb{Z}/m\mathbb{Z}$ factors through $H_1(\Gamma, \mathbb{Z}/m\mathbb{Z})$. The factorization $H_1(\Gamma, \mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{Z}/m\mathbb{Z}$ is the map φ_γ from the introduction. In other words, the following diagram commutes.

$$\begin{array}{ccc} \pi_1(\Gamma, v) & \xrightarrow{(\vec{x}, \sigma)} & S(m, n) \\ \downarrow & & \downarrow \eta_{m,m} \\ H_1(\Gamma, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\varphi_\gamma} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

4. PROOF OF THEOREMS 2 AND 3

The goal of this section is to prove Theorems 2 and 3. Throughout, we assume that (Γ, γ) is a 2-vertex connected twist graph with $n + 1$ vertices, no loops, and with φ_γ surjective.

Let v be a vertex in Γ . For ease of notation, we write G for the image of $\pi_1(\Gamma, v)$ in $S(m, n)$ and H for $G \cap \ker(\pi)$. To classify the possible subgroups $G \subseteq S(m, n)$, we begin by classifying the possible subgroups $H \subseteq (\mathbb{Z}/m\mathbb{Z})^n$. To that end, we first identify a useful subset of H . The existence of this subset is essentially the only part of the argument that uses the homomorphism from $\pi_1(\Gamma, v)$ to $S(m, n)$.

Proposition 1. *There exists a set of generators $\{a_1, \dots, a_g\} \subseteq \mathbb{Z}/m\mathbb{Z}$ and vectors $\vec{x}_1, \dots, \vec{x}_g \in H$ such that, for each vertex $w \in V(\Gamma)$, $\vec{x}_{i,w}$ is either 0 or a_i . Moreover, there is at least one vertex w such that $\vec{x}_{i,w} = a_i$, and, if Γ is not a cycle (possibly with multi-edges), then there is at least one vertex w such that $\vec{x}_{i,w} = 0$.*

Proof. Let p_1, \dots, p_g be the simple closed paths from Lemma 1. Let

$$a_i = \varphi_\gamma(p_i) = \eta_{m,m}(\vec{x}(p_i), \sigma_{p_i}) = \sum_{w \in V(\Gamma)} \vec{x}(p_i)_w \pmod{m}.$$

If w is a vertex of Γ not contained in p_i , then $\vec{x}(p_i)_w = 0$. Thus, the sum on the right can be taken over vertices in p_i . Since the closed paths p_i generate $\pi_1(\Gamma, v)$ and φ_γ is surjective, the elements a_1, \dots, a_g generate $\mathbb{Z}/m\mathbb{Z}$.

Suppose that p_i has length k_i . Then σ_{p_i} is a $(k_i - 1)$ -cycle. Thus,

$$(\vec{x}(p_i), \sigma_{p_i})^{k_i-1} = \left(\sum_{j=0}^{k_i-2} \sigma_{p_i}^j \vec{x}(p_i) \sigma_{p_i}^{-j}, \text{id} \right).$$

Therefore, there exists a vector \vec{x}_i such that $(\vec{x}(p_i), \sigma_{p_i})^{k_i-1} = (\vec{x}_i, \text{id}) \in G \cap \ker(\pi) = H$. If w is a vertex of Γ not contained in p_i , then $\vec{x}_{i,w} = 0$. On the other hand, if w is contained in p_i , then

$$\vec{x}_{i,w} \equiv \sum_{j=1}^{k_i-1} \vec{x}(p_i)_{\sigma_{p_i}^j(w)} \equiv \sum_{u \in p_i} \vec{x}(p_i)_u \equiv a_i \pmod{m}.$$

Finally, if Γ is not a cycle (possibly with multi-edges), then the paths p_i can be chosen to be non-Hamiltonian. Thus, there exists a vertex w not contained in p_i , and the result follows. \square

Note that $H \subseteq (\mathbb{Z}/m\mathbb{Z})^n$ is invariant under the action of $\pi(G)$. In [5], Mortimer studies subspaces of a vector space k^n that are invariant under known 2-transitive subgroups of S_n . If m is prime, we can use Mortimer's result to classify the possibilities for H . More generally, for any integer a dividing m , let $G_a = \rho_{m,a}(G)$ and $H_a = \rho_{m,a}(H)$. We write

$$\mathcal{C}_{m,a}^\perp := \ker(\eta_{m,a}) \cap \ker(\pi) = \left\{ \vec{x} \in (\mathbb{Z}/m\mathbb{Z})^n \mid \sum_{i=1}^n x_i \equiv 0 \pmod{a} \right\}.$$

Proposition 2. *If Γ is not a cycle (possibly with multi-edges) and p is a prime dividing m , then H_p is equal to either $(\mathbb{Z}/p\mathbb{Z})^n$ or $\mathcal{C}_{p,p}^\perp$.*

Proof. The group H_p is invariant under the action of $\pi(G)$, and by Theorem 1, $\pi(G)$ is either S_n ($n \geq 3$), A_n ($n \geq 4$), or $\text{PGL}(2, 5)$ ($n = 6$). By [5, Table 1], the heart over $\mathbb{Z}/p\mathbb{Z}$ of $\pi(G)$ acting on $(\mathbb{Z}/p\mathbb{Z})^n$ is simple². Note also that a $(\mathbb{Z}/p\mathbb{Z})$ -subspace of $(\mathbb{Z}/p\mathbb{Z})^n$ is the same thing as a subgroup of $(\mathbb{Z}/p\mathbb{Z})^n$. Thus, by [5, Lemma 2], the only $\pi(G)$ -invariant subgroups of $(\mathbb{Z}/p\mathbb{Z})^n$ are 0, the group of constant vectors $\mathcal{C}_{p,p}$, $\mathcal{C}_{p,p}^\perp$, and $(\mathbb{Z}/p\mathbb{Z})^n$.

We now prove that H_p cannot be 0 or $\mathcal{C}_{p,p}$. By Proposition 1, there exists a set of generators $\{a_1, \dots, a_g\} \subseteq \mathbb{Z}/m\mathbb{Z}$ and vectors $\vec{x}_1, \dots, \vec{x}_g \in H$ such that, for each vertex $w \in V(\Gamma)$, $\vec{x}_{i,w}$ is either 0 or a_i . Moreover, if $a_i \neq 0$, then the vector \vec{x}_i is non-constant. Since the set $\{a_1, \dots, a_g\}$ generates $\mathbb{Z}/m\mathbb{Z}$, there exists an i such that a_i is not divisible by p . Thus, $\rho_{m,p}(\vec{x}_i)$ is non-constant, and it follows that H_p cannot be 0 or $\mathcal{C}_{p,p}$. \square

In addition, the only primes p for which H_p may be equal to $\mathcal{C}_{p,p}^\perp$ are $p = 2$ and 3, and these two cases are mutually exclusive.

Lemma 4. *Let $a > 1$ be an integer dividing m . If $H = \mathcal{C}_{m,a}^\perp$, then either $a = 2$ and $\pi(G)$ is S_n or $\text{PGL}(2, 5)$, or $a = 3$ and $\pi(G)$ is A_4 .*

²Indeed, the heart over any field k of $\pi(G)$ is simple, except for the special case where $\pi(G) = A_4$. The heart of A_4 over k is reducible if k contains the field \mathbb{F}_4 , but this does not occur in our case.

Proof. By assumption, the map $\eta_{m,a}$ is surjective. If $H = \mathcal{C}_{m,a}^\perp$, then the restriction of $\eta_{m,a}$ to H is identically zero, and thus $\eta_{m,a}$ factors through the quotient $G/H \cong \pi(G)$. If $\pi(G)$ is A_n for $n \geq 5$, then since A_n is simple, the map $A_n \rightarrow \mathbb{Z}/a\mathbb{Z}$ is trivial, contradicting the fact that $\eta_{m,a}$ is surjective.

If $\pi(G) = A_4$, then since all the subgroups of order 3 in A_4 are conjugate, no proper normal subgroup can contain an element of order 3. It follows that the only non-trivial proper normal subgroup of A_4 is the Klein 4-group K , and the surjective map $A_4 \rightarrow \mathbb{Z}/a\mathbb{Z}$ must be the quotient $q : A_4 \rightarrow A_4/K \cong \mathbb{Z}/3\mathbb{Z}$.

Similarly, if $\pi(G)$ is S_n or $\mathrm{PGL}(2,5) \cong S_5$, then a must be 2 and the map $S_n \rightarrow \mathbb{Z}/a\mathbb{Z}$ must be the sign homomorphism. To see this, note that every transposition in S_n must map to either the identity or an element of order 2. Since the transpositions generate S_n , it follows that every element of S_n must map to either the identity or an element of order 2. Since $\mathbb{Z}/a\mathbb{Z}$ contains at most one element of order 2, the surjectivity of $\eta_{m,a}$ implies that $a = 2$. Finally, since the transpositions are all conjugate in S_n and $\mathbb{Z}/2\mathbb{Z}$ is abelian, we see that all the transpositions must map to the same element of $\mathbb{Z}/2\mathbb{Z}$. It follows that all odd permutations must map to 1 and all even permutations to 0. \square

We now classify the possible subgroups H .

Proposition 3. *Let a divide m . If Γ is not a cycle (possibly with multi-edges), then either $H_a = (\mathbb{Z}/a\mathbb{Z})^n$, or $H_a = \mathcal{C}_{a,2}^\perp$ and $\pi(G)$ is S_n or $\mathrm{PGL}(2,5)$, or $H_a = \mathcal{C}_{a,3}^\perp$ and $\pi(G)$ is A_4 .*

Proof. We will prove this by induction on the number of prime factors of a . The base case, where $a = p$ is prime, follows from Proposition 2 and Lemma 4.

We break the inductive step into two cases. First, consider the case where $a = a_1 a_2$, where $a_1, a_2 > 1$ are relatively prime. If $\pi(G) = S_n$ or $\pi(G) = \mathrm{PGL}(2,5)$, assume without loss of generality that a_2 is odd. Similarly, if $\pi(G) = A_4$, assume that a_2 is not divisible by 3. By the Chinese Remainder Theorem, $(\mathbb{Z}/a\mathbb{Z})^n \cong (\mathbb{Z}/a_1\mathbb{Z})^n \times (\mathbb{Z}/a_2\mathbb{Z})^n$. By induction, $H_{a_2} = (\mathbb{Z}/a_2\mathbb{Z})^n$, hence $H_a = \rho_{a,a_1}^{-1}(H_{a_1})$. By induction, H_{a_1} is either $(\mathbb{Z}/a_1\mathbb{Z})^n$ or $\mathcal{C}_{a_1,p}^\perp$ for $p = 2$ or 3 . If $H_{a_1} = (\mathbb{Z}/a_1\mathbb{Z})^n$, we see that $H_a = (\mathbb{Z}/a\mathbb{Z})^n$, and if $H_{a_1} = \mathcal{C}_{a_1,p}^\perp$, we see that $H_a = \mathcal{C}_{a,p}^\perp$.

Second, consider the case where $a = p^\alpha$ is a prime power, with $\alpha > 1$. Consider the short exact sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^n \xrightarrow{\mu} (\mathbb{Z}/p^\alpha\mathbb{Z})^n \xrightarrow{r} (\mathbb{Z}/p^{\alpha-1}\mathbb{Z})^n \longrightarrow 0,$$

where the map μ is multiplication by $p^{\alpha-1}$ and the map r is reduction $(\bmod p^{\alpha-1})$. By induction, $r(H_a) = H_{p^{\alpha-1}}$ is either $(\mathbb{Z}/p^{\alpha-1}\mathbb{Z})^n$ or $\mathcal{C}_{p^{\alpha-1},p}^\perp$. Moreover, $\mu^{-1}(H_a)$ is invariant under the action of $\pi(G)$, so by [5, Lemma 2] it is either 0, $\mathcal{C}_{p,p}$, $\mathcal{C}_{p,p}^\perp$, or $(\mathbb{Z}/a\mathbb{Z})^n$. We consider each of these four possibilities in turn.

If $\mu^{-1}(H_a) = (\mathbb{Z}/a\mathbb{Z})^n$, then $H_a = r^{-1}(H_{p^{\alpha-1}})$. If $H_{p^{\alpha-1}} = (\mathbb{Z}/p^{\alpha-1}\mathbb{Z})^n$, then $H_a = (\mathbb{Z}/a\mathbb{Z})^n$, and if $H_{p^{\alpha-1}} = \mathcal{C}_{p^{\alpha-1},p}^\perp$, then $H_a = \mathcal{C}_{p^\alpha,p}^\perp$.

Now, suppose that $\mu^{-1}(H_a) = \mathcal{C}_{p,p}$, and let $\vec{x} \in H_a$ be an element such that $r(\vec{x}) = (p^{\alpha-2}, (p-1)p^{\alpha-2}, 0, 0, \dots, 0)$. Then $p\vec{x}$ is in the image of μ , but not in $\mu(\mathcal{C}_{p,p})$, a contradiction.

Similarly, if $\mu^{-1}(H_a) = 0$, then since $r(H_a)$ is nonzero, there exists a nonzero $\vec{x} \in H_a$. Consider the smallest power β such that $p^\beta \vec{x}$ is in the image of μ . Note

that such a β must exist, since $p^{\alpha-1}\vec{x}$ is in the image of μ for all \vec{x} . Since $\vec{x} \neq 0$, we see that $p^\beta \vec{x} \neq 0$, a contradiction.

Finally, suppose that $\mu^{-1}(H_a) = \mathcal{C}_{p,p}^\perp$. If either $\alpha \geq 3$ or $H_{p^{\alpha-1}} = (\mathbb{Z}/p^{\alpha-1}\mathbb{Z})^n$, then there exists a vector $\vec{x} \in H_a$ such that $r(\vec{x}) = (p^{\alpha-2}, 0, 0, \dots, 0)$. Then $p\vec{x}$ is in the image of μ , but not in $\mu(\mathcal{C}_{p,p}^\perp)$, a contradiction. It follows that $a = p^2$ and $H_p = \mathcal{C}_{p,p}^\perp$. We now show that this is impossible.

Consider the case where $p = 2$ and $\pi(G) = S_n$ or $\text{PGL}(2, 5)$. We will show that $H_4 \subseteq \mathcal{C}_{4,4}^\perp$, contradicting Lemma 4. Assume for contradiction that there exists $\vec{x} \in H_4$ with $\eta_{4,4}(\vec{x}) \not\equiv 0 \pmod{4}$. Since $H_2 = \mathcal{C}_{2,2}^\perp$, we see that $\eta_{4,4}(\vec{x}) \equiv 2 \pmod{4}$. Let

$$\mathcal{O} = \{w \in V(\Gamma) \setminus \{v\} \mid \vec{x}_w \text{ is odd}\}.$$

We reduce to the case where \mathcal{O} is strictly contained in $V(\Gamma) \setminus \{v\}$. Since the set $\{a_1, \dots, a_g\}$ generates $\mathbb{Z}/2\mathbb{Z}$, there exists an i such that a_i is odd. If $\eta_{m,4}(\vec{x}_i) \not\equiv 0 \pmod{4}$, then we let \vec{x} be $\rho_{m,4}(\vec{x}_i)$. If $\mathcal{O} = V(\Gamma) \setminus \{v\}$ and $\eta_{m,4}(\vec{x}_i) \equiv 0 \pmod{4}$, then replace \vec{x} with $\vec{x} + \rho_{m,4}(\vec{x}_i)$.

Now, let $A \subset V(\Gamma) \setminus \{v\}$ be a subset containing \mathcal{O} of size $|A| = k$, with k odd. (If $\pi(G) = \text{PGL}(2, 5)$, choose $k = 5$.) Let $\sigma \in \pi(G)$ be a k -cycle that transitively permutes the elements of A . (In the case where $\pi(G) = \text{PGL}(2, 5)$, there is a 5-cycle in $\text{PGL}(2, 5)$ fixing any given element of $\mathbb{P}^1(\mathbb{F}_5)$.) Since H_4 is $\pi(G)$ -invariant, it contains the vector $\vec{z} = \sum_{i=0}^{k-1} \sigma^i \cdot \vec{x}$. Note that

$$\vec{z}_w = \begin{cases} \sum_{u \in A} \vec{x}_u & \text{if } w \in A, \\ k\vec{x}_w & \text{if } w \notin A. \end{cases}$$

Since $\sum_{u \in A} \vec{x}_u$ is even, \vec{z} is in the image of μ . On the other hand, we have

$$\eta_{4,4}(\vec{z}) \equiv k \left(\sum_{u \in A} \vec{x}_u + \sum_{u \notin A} \vec{x}_u \right) \equiv k\eta_{4,4}(\vec{x}) \equiv 2 \pmod{4},$$

where the last equality holds because k is odd. Thus, \vec{z} is in the image of μ , but not in $\mu(\mathcal{C}_{2,2}^\perp)$, a contradiction.

Similarly, consider the case where $p = 3$ and $\pi(G) = A_4$. As above, there exists an i such that a_i is not divisible by 3. The vector $\rho_{m,9}(\vec{x}_i)$ must have one entry equal to 0 and three entries equal to $a_i \pmod{3}$. Let $K \subset A_4$ be the Klein 4-group. Since H_9 is A_4 -invariant, it contains the vector $\vec{z} = \sum_{\sigma \in K} \sigma \cdot \rho_{m,9}(\vec{x}_i) = (3a_i, 3a_i, 3a_i, 3a_i)$. Thus, \vec{z} is in the image of μ , but not in $\mu(\mathcal{C}_{3,3}^\perp)$, a contradiction. \square

We now prove the main theorem.

Proof of Theorem 2. If Γ is bipartite, then by Theorem 1, $\pi(G) = A_n$. The only simple 2-vertex connected bipartite graphs on 5 or fewer vertices are the 4-cycle and the graph Θ_5 . By assumption, therefore, $n \geq 5$. By Proposition 3, $H = (\mathbb{Z}/m\mathbb{Z})^n$. It follows that

$$G = \pi^{-1}(A_n) = \left\{ (\vec{x}, \sigma) \in S(m, n) \mid \sigma \in A_n \right\}.$$

If Γ is not bipartite, then by Theorem 1, $\pi(G) = S_n$ with $n \geq 3$. By Proposition 3, either $H = (\mathbb{Z}/m\mathbb{Z})^n$ or m is even and $H = \mathcal{C}_{m,2}^\perp$. If $H = (\mathbb{Z}/m\mathbb{Z})^n$, then

$$G = \pi^{-1}(S_n) = S(m, n).$$

If $H = \mathcal{C}_{m,2}^\perp$, then as in the proof of Lemma 4, the map $\eta_{m,2} : G \rightarrow \mathbb{Z}/2\mathbb{Z}$ factors through the quotient $G/H \cong S_n$, and the map $S_n \rightarrow \mathbb{Z}/2\mathbb{Z}$ is the sign homomorphism. It follows that

$$G = \left\{ (\vec{x}, \sigma) \in S(m, n) \mid \eta_{m,2}(\vec{x}, \sigma) \equiv \text{sign}(\sigma) \pmod{2} \right\}.$$

It remains to classify which non-bipartite twist graphs correspond to each of the two subgroups. If p is a simple closed path, then $\eta_{m,2}(x(\vec{p}), \sigma_p) \equiv \text{sign}(\sigma_p) \pmod{2}$ if and only if p has an even number of edges e with γ_e even. Since $\pi_1(\Gamma, v)$ is generated by simple closed paths, it follows that $G \neq S(m, n)$ if and only if (Γ, γ) is twist bipartite. \square

The exceptional cases are very similar.

Proof of Theorem 3. If $\Gamma = \Theta_7$, the argument is nearly identical to the non-bipartite case above. By Theorem 1, $\pi(G) = \text{PGL}(2, 5)$, and by Proposition 3, either $H = (\mathbb{Z}/m\mathbb{Z})^6$ or m is even and $H = \mathcal{C}_{m,2}^\perp$. If $H = (\mathbb{Z}/m\mathbb{Z})^6$, then $G = \pi^{-1}(\text{PGL}(2, 5))$, and if $H = \mathcal{C}_{m,2}^\perp$, then as in the proof of Lemma 4, the map $\eta_{m,2} : G \rightarrow \mathbb{Z}/2\mathbb{Z}$ factors through the quotient $G/H \cong \text{PGL}(2, 5) \cong S_5$, and the map $S_5 \rightarrow \mathbb{Z}/2\mathbb{Z}$ is the sign homomorphism.

If $\Gamma = \Theta_5$, then by Proposition 3, either $H = (\mathbb{Z}/m\mathbb{Z})^4$ or m is divisible by 3 and $H = \mathcal{C}_{m,3}^\perp$. If $H = (\mathbb{Z}/m\mathbb{Z})^4$, then again

$$G = \pi^{-1}(A_4) = \left\{ (\vec{x}, \sigma) \in S(m, 4) \mid \sigma \in A_4 \right\}.$$

If $H = \mathcal{C}_{m,3}^\perp$, then as in the proof of Lemma 4, the map $\eta_{m,3} : G \rightarrow \mathbb{Z}/3\mathbb{Z}$ factors through the quotient $G/H \cong A_4$, and the map $A_4 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is the quotient of A_4 by the Klein 4-group K . It follows that

$$G = \left\{ (\vec{x}, \sigma) \in S(m, 4) \mid \sigma \in A_4, \eta_{m,3}(\vec{x}, \sigma) \equiv q(\sigma) \pmod{3} \right\}.$$

It remains to classify which 1-chains γ on Θ_4 correspond to each of these two subgroups. The fundamental group of Θ_4 is generated by the two simple closed paths $p = e_1 e_2 \bar{e}_6 \bar{e}_5$ and $q = e_3 e_4 \bar{e}_6 \bar{e}_5$. Because these two paths generate $\pi_1(\Gamma, v)$, G is contained in a subgroup if and only if $(\vec{x}(p), \sigma_p)$ and $(\vec{x}(q), \sigma_q)$ are contained in that subgroup. It follows that $G \neq \pi^{-1}(A_4)$ if and only if the two classes

$$(\gamma_{e_1} + \gamma_{e_2}) - (\gamma_{e_5} + \gamma_{e_6}), \quad (\gamma_{e_3} + \gamma_{e_4}) - (\gamma_{e_5} + \gamma_{e_6})$$

are nonzero and distinct $\pmod{3}$. Equivalently, $G \neq \pi^{-1}(A_4)$ if and only if

$$(\gamma_{e_1} + \gamma_{e_2}), \quad (\gamma_{e_3} + \gamma_{e_4}), \quad \text{and} \quad (\gamma_{e_5} + \gamma_{e_6})$$

are distinct $\pmod{3}$.

Finally, note that for any three integers x, y , and z , we have $x+y+z \equiv 0 \pmod{3}$ if and only if either x, y , and z are all equivalent $\pmod{3}$, or distinct $\pmod{3}$. But if

$$(\gamma_{e_1} + \gamma_{e_2}) \equiv (\gamma_{e_3} + \gamma_{e_4}) \equiv (\gamma_{e_5} + \gamma_{e_6}) \pmod{3},$$

then φ_γ is not surjective. It follows that $G \neq \pi^{-1}(A_4)$ if and only if $\sum_{i=1}^6 \gamma_{e_i} \equiv 0 \pmod{3}$. \square

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