TROPICAL MULTIPLICATION MAPS AND THE GIESEKER-PETRI THEOREM

DAVID JENSEN AND SAM PAYNE

ABSTRACT. We develop a framework to apply tropical and nonarchimedean analytic techniques to multiplication maps on linear series and study degenerations of these multiplications maps when the special fiber is not of compact type. As an application, we give a tropical criterion for a curve over a valued field to be Gieseker-Petri general.

1. INTRODUCTION

The schemes $\mathcal{G}_d^r(X)$ parameterizing linear series of degree d and dimension r on a smooth curve X of genus g are the basic objects of study in classical Brill-Noether theory. The Brill-Noether number $\rho(g, r, d) = g - (r + 1)(g - d + r)$ is a naive dimension estimate for $\mathcal{G}_d^r(X)$, and the following two fundamental results give the local structure of these schemes when X is general in its moduli space.

Brill-Noether Theorem. [GH80] Let X be a general curve of genus g. Then $\mathcal{G}_d^r(X)$ has pure dimension $\rho(g, r, d)$ if this is nonnegative and is empty otherwise.

Gieseker-Petri Theorem. [Gie82] Let X be a general curve of genus g. Then $\mathcal{G}_d^r(X)$ is smooth.

The dimension of the Zariski tangent space to $\mathcal{G}_d^r(X)$ at a linear series $W \subset \mathcal{L}(D)$ is equal to $\rho(g, r, d)$ plus the dimension of the kernel of the multiplication map

$$\mu_W : W \otimes \mathcal{L}(K_X - D) \to \mathcal{L}(K_X).$$

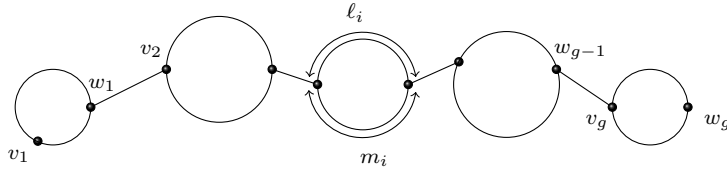In particular, $\mathcal{G}_d^r(X)$ is smooth of dimension $\rho(g, r, d)$ at a linear series W if and only if the multiplication map μ_W is injective [ACGH85, §IV.4].

Gieseker's original proof that μ_W is injective for all W when X is general involves a subtle degeneration argument. Eisenbud and Harris developed a more systematic method for studying limits of linear series for one-parameter degenerations of curves in which the special fiber has compact type, and applied this theory to give a simpler proof of the Gieseker-Petri Theorem [EH83, EH86]. Another proof, using vector bundles on K3 surfaces instead of degenerations, is due to Lazarsfeld [Laz86].

Here we use degeneration techniques of a different nature, studying limits of linear series over a one-parameter degeneration of curves where the special fiber is not of compact type. These techniques are based in tropical geometry and Berkovich's theory of nonarchimedean analytic curves and their skeletons. Our starting point is the tropical proof of the Brill-Noether Theorem [CDPR12]. At the end of the day, we arrive at a similar tropical condition for a curve X over a valued field to be Gieseker-Petri general, having the property that the multiplication maps μ_W are injective for all linear series W on X.

Let Γ be a chain of g loops connected by bridges, with generic edge lengths.

FIGURE 1. The graph Γ.

Here the genericity condition on edge lengths is the same as in [CDPR12]; we require that ℓ_i/m_i is not equal to the ratio of two positive integers whose sum is less than or equal to $2g - 2$.

Theorem 1.1. *Let X be a smooth projective curve of genus g over a complete nonarchimedean field such that the skeleton of the Berkovich analytic space X^{an} is isometric to Γ. Then the multiplication map*

$$\mu_W : W \otimes \mathcal{L}(K_X - D) \to \mathcal{L}(K_X)$$

is injective for all linear series $W \subset \mathcal{L}(D)$ on X.

There do exist such curves over valued fields of arbitrary pure or mixed characteristic. This follows from the fact that the moduli space of tropical curves is the skeleton of the Deligne-Mumford compactification of the moduli space of curves [ACP12], and can also be proved by deformation theory, as in [Bak08, Appendix B]. The existence of Gieseker-Petri general curves over an arbitrary algebraically closed field then follows by standard arguments from scheme theory, using the fact that the coarse moduli space of curves is defined over $\operatorname{Spec} \mathbb{Z}$, as in [CDPR12, Section 3].

Remark 1.2. The tropical Jacobians of two graphs that differ by the addition or deletion of a bridge are canonically isomorphic, and these isomorphisms respect the images of the Abel-Jacobi maps, so the Brill-Noether theory of Γ is the same as that of the chain of loops without bridges studied in [CDPR12]. See [LPP12, Len12] for the basics of tropical Brill-Noether theory and further references.

We now briefly sketch relations between the approach developed here, the classical theory of limit linear series, and the tropical theory of divisors on graphs.

Suppose X is defined over a discretely valued field with valuation ring R, and let L be a line bundle on X. Consider a regular model $\mathcal{X}$ over $\operatorname{Spec} R$ with general fiber X, in which the special fiber $\overline{\mathcal{X}}$ is semistable with smooth components $\overline{\mathcal{X}}_i$. The special fiber has compact type, meaning that its Jacobian is compact, if and only if its dual graph is a tree. In this case, for each component $\overline{\mathcal{X}}_i$ there is a unique extension $\mathcal{L}_i$ of the line bundle L such that

$$\deg\left(\mathcal{L}_i|_{\overline{\mathcal{X}}_j}\right) = \begin{cases} d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Given a linear subspace $W \subset H^0(X, L)$ of degree d and dimension $r + 1$, the R-submodule $\mathcal{W}_i \subset W$ consisting of sections that extend to $\mathcal{L}_i$ is free of rank $r + 1$, and restricts to a linear series of degree d and dimension r on $\overline{\mathcal{X}}_i$. The theory of limit linear series studies these distinguished linear series on the components of the special fiber, with special attention to their vanishing sequences at the nodes of $\overline{\mathcal{X}}$.

In contrast, if $\overline{\mathcal{X}}$ is not of compact type, then its dual graph is not a tree, and there is an obstruction to extending L to a line bundle $\mathcal{L}_i$ with degrees as above on

the components of the special fiber. This obstruction is given by an element in the component group of the Néron model of the Jacobian of X.

The theory of divisors on graphs follows a deep analogy between divisors on algebraic curves and the distributions of degrees of specializations of L over the components of the special fiber. In this framework, one considers the dual graph whose vertices v_i correspond to components $\overline{\mathcal{X}}_i$ and whose edges correspond to nodes of $\overline{\mathcal{X}}$. Then an extension $\mathcal{L}$ of L to $\mathcal{X}$ gives rise to a formal sum

$$D_{\mathcal{L}} = \sum_i \deg(\mathcal{L}|_{\overline{\mathcal{X}}_i}) v_i,$$

which is considered as a divisor on the graph. Since the divisors arising from different specializations of L differ by a sequence of chip-firing moves, one studies the tropical Picard group parametrizing equivalence classes of divisors on the graph modulo the relation generated by chip-firing. The tropical Jacobian, the degree zero part of this tropical Picard group, is canonically identified with the component group of the Néron model of the Jacobian of X.

Baker's Specialization Lemma [Bak08] says that a line bundle whose complete linear series has dimension r can be specialized so that all degrees are nonnegative and the distribution of degrees dominates any given divisor of degree r on the dual graph. In other words, it has rank at least r in the sense of [BN07]. Therefore, the specialization of any line bundle whose complete linear series has dimension at least r lies in the tropical Brill-Noether locus parametrizing divisor classes of degree d with rank at least r. In [CDPR12], a careful analysis of the Brill-Noether loci of the chain of loops shows that if a curve X has a regular semistable model whose special fiber has this dual graph, then the curve must be Brill-Noether general, meaning that $\mathcal{G}_d^r(X)$ has dimension $\rho(g, r, d)$ if this is positive, and is empty otherwise. In particular, we get not only a new proof of the Brill-Noether Theorem, but an explicit and computationally verifiable sufficient condition for a curve to be Brill-Noether general, the existence of a regular semistable model whose special fiber has a particular dual graph. Similar arguments, combined with the basepoint-free pencil trick, lead to a proof of the Gieseker-Petri Theorem in the special case where $r = 1$ [BJM$^+$12].

Remark 1.3. This tropical proof of the Brill-Noether Theorem can be reframed in the language of Berkovich's nonarchimedean analytic geometry to show that any curve of genus g over a valued field whose skeleton is a chain of g loops with generic edge lengths must be Brill-Noether general. In the body of the paper, we follow this more general approach, with skeletons of analytifications in place of dual graphs of regular semistable models.

In the present work, we go beyond classical tropical methods, which only see the possible distribution of degrees over the components of the special fiber, in order to work with one-parameter degenerations of multiplication maps in the case where the special fiber does not have compact type. The required machinery involves not only the piecewise linear geometry of skeletons, but also algebraic geometry over the residue field. In particular, we use Thuillier's nonarchimedean analytic Poincaré-Lelong formula [Thu05, BPR11], which relates orders of vanishing at nodes in the special fiber of a semistable model to slopes of piecewise linear functions on the skeleton. Nevertheless, at the end of the day, we obtain the tropical criterion for a curve over a valued field to be Gieseker-Petri general stated in Theorem 1.1, depending only on the metric graph properties of the skeleton.

Remark 1.4. The reason that we consider a graph with bridges between the loops is to use the slopes of piecewise linear functions along these edges. These are related to orders of vanishing along separating nodes in the special fiber by the nonarchimedean analytic Poincaré-Lelong formula.

Remark 1.5. In some ways, the tropical geometry of divisors on the chain of loops Γ is similar to the geometry of limit linear series on a chain of elliptic curves with generic attaching points. As is well-known to experts in Brill-Noether theory, the theory of limit linear series on such curves gives a characteristic-free proof of the Brill-Noether and Gieseker-Petri theorems [Oss11, CLMTiB12]. Some parts of our approach, including Lemma 7.2 and Proposition 7.4, can be viewed as tropical analogues of facts about limit linear series on chains of elliptic curves. However, there appears to be no direct dictionary between the two. Our arguments take advantage of combinatorial considerations that seem to be invisible from the perspective of limit linear series. In the limit linear series proofs of the Gieseker-Petri Theorem, both [EH83] and [CLMTiB12] assume the multiplication map is not injective and use a degeneration argument to construct a divisor in $|K_X|$ of *impossible degree*. We assume the multiplication map is not injective and reach a contradiction by constructing an impossible divisor in $|K_\Gamma|$, but it is not the degree of this divisor that creates the contradiction. Our argument relies on Proposition 3.5 and Lemma 4.3 to show that the divisor has *impossible shape*.

2. Background

We briefly review the theory of divisors and divisor classes on metric graphs, along with relations to the classical theory of algebraic curves via Berkovich analytification and specialization to skeletons. For further details and references, see [BN07, Bak08, BPR11, AB12].

2.1. **Divisors on graphs and Riemann-Roch.** Let Γ be a metric graph. A *divisor* on Γ is a finite formal sum

$$D = a_1 v_1 + \cdots + a_s v_s,$$

where the v_i are points in Γ and the coefficients a_i are integers. The *degree* of a divisor is the sum of its coefficients

$$\deg(D) = a_1 + \cdots + a_s,$$

and a divisor is *effective* if all of its coefficients are nonnegative. We say that an effective divisor *contains* a point v_i if its coefficient a_i is strictly positive. We will frequently consider questions about whether a given effective divisor D contains at least one point in a connected subset $\Gamma' \subset \Gamma$. See, for instance, Section 3.2.

Let $\mathrm{PL}(\Gamma)$ be the additive group of continuous piecewise linear functions ψ with integer slopes on Γ. (Throughout, all of the piecewise linear functions that we consider have integer slopes.) The *order* of such a piecewise linear function ψ at a point v is the sum of its incoming slopes along edges containing v, and is denoted

$\mathrm{ord}_v(\psi)$. Note that $\mathrm{ord}_v(\psi)$ is zero for all but finitely many points v in Γ, so

$$\mathrm{div}(\psi) = \sum_{v \in \Gamma} \mathrm{ord}_v(\psi)\, v$$

is a divisor. A divisor is *principal* if it is equal to $\mathrm{div}(\psi)$ for some piecewise linear function ψ, and two divisors D and D' are *equivalent* if $D - D'$ is principal. Note that every principal divisor has degree zero, so the group $\mathrm{Pic}(\Gamma)$ of equivalence classes of divisors is graded by degree.

Let D be a divisor on Γ. The *complete linear series* $|D|$ is the set of effective divisors on Γ that are equivalent to D, and

$$R(D) = \{\psi \in \mathrm{PL}(\Gamma) \mid D + \mathrm{div}(\psi) \text{ is effective}\}.$$

These objects are closely analogous to the complete linear series of a divisor on an algebraic curve, and the vector space of rational functions with poles bounded by that divisor. There is a natural surjective map from $R(D)$ to $|D|$ taking a piecewise linear function ψ to $\mathrm{div}(\psi) + D$, and two functions ψ and ψ' have the same image in $|D|$ if and only if $\psi - \psi'$ is constant. The vector space structure on rational functions with bounded poles is analogous to the *tropical module* structure on $R(D)$. Addition in this tropical module is given by the pointwise minimum; if $\psi_0, \ldots, \psi_r$ are in $R(D)$ and $b_0, \ldots, b_r$ are real numbers, then the function θ given by

$$\theta(v) = \min_j \left\{\psi_j(v) + b_j\right\}$$

is also in $R(D)$ [HMY12].

The *rank* $r(D)$ is the largest integer r such that $D - E$ is equivalent to an effective divisor for every effective divisor E of degree r. In other words, a divisor D has rank at least r if and only if its linear series contains divisors that dominate any effective divisor of degree r. This invariant satisfies the following Riemann-Roch theorem with respect to the *canonical divisor* $K_\Gamma = \sum_{v \in \Gamma}(\deg(v) - 2)\, v$.

Tropical Riemann-Roch Theorem. [BN07, GK08, MZ08] *Let D be a divisor on a metric graph Γ with first Betti number g. Then*

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1.$$

2.2. **Specialization of divisors from curves to graphs.** Throughout, we work over a fixed algebraically closed field K that is complete with respect to a nontrivial valuation

$$\mathrm{val} : K^* \to \mathbb{R}.$$

Let $R \subset K$ be the valuation ring, and let κ be the residue field.

Let X be an algebraic curve over K. The underlying set of the Berkovich analytic space X^{an} consists of the closed points $X(K)$ together with the set of valuations on the function field $K(X)$ that extend the given valuation on K. We write

$$\mathrm{val}_y : K(X)^* \to \mathbb{R}$$

for the valuation corresponding to a point y in $X^{\mathrm{an}} \smallsetminus X(K)$. The topology on X^{an} is the weakest containing U^{an} for every Zariski open U in X and such that, for any $f \in \mathcal{O}_X(U)$, the function taking $x \in U(K)$ to $\mathrm{val}(f(x))$ and $y \in U^{\mathrm{an}} \smallsetminus U(K)$ to $\mathrm{val}_y(f)$ is continuous.

The points in $X(K)$ are called type-1 points of X^{an}, and the remaining points in $X^{\mathrm{an}} \smallsetminus X(K)$ are classified into three more types according to the algebraic properties of the corresponding valuation. For our purposes, the most relevant points are type-2 points, the points y such that the residue field of $K(X)$ with respect to val_y has

transcendence degree 1 over κ. We write X_y for the smooth projective curve over
the residue field of K with this function field.

Remark 2.1. By passing to a spherically complete extension field whose valuation
surjects onto $\mathbb{R}$, one could assume that all points in $X^{\mathrm{an}} \smallsetminus X(K)$ are of type-2.

Suppose X is smooth and projective. Then X has a *semistable vertex set*, a
finite set of type-2 points whose complement is a disjoint union of a finite number
of open annuli and an infinite number of open balls. Each semistable vertex set
$V \subset X^{\mathrm{an}}$ corresponds to a semistable model $\mathcal{X}_V$ of X. The normalized irreducible
components of the special fiber $\overline{\mathcal{X}}_V$ are naturally identified with the curves X_y, for
$y \in V$, and the preimages of the nodes in $\overline{\mathcal{X}}_V$ under specialization are the annuli
in $X^{\mathrm{an}} \smallsetminus V$. The annulus corresponding to a node where X_y meets $X_{y'}$ contains
a unique embedded open segment with endpoints y and y', whose length is the
logarithmic modulus of the annulus. The union of these open segments together
with V is a closed connected metric graph embedded in $X^{\mathrm{an}} \smallsetminus X(K)$ with a natural
metric. We write Γ_V for this metric graph, and call it the *skeleton* of the semistable
model $\mathcal{X}_V$. If X has genus at least 2, which we may assume since the Gieseker-Petri
Theorem is trivial for curves of genus 0 and 1, there is a unique minimal semistable
vertex set in X^{an}. We write Γ for the skeleton of this minimal semistable vertex
set, and call it simply the *skeleton* of X^{an}.

Each connected component of $X^{\mathrm{an}} \smallsetminus \Gamma$ has a unique boundary point in Γ, and
there is a canonical retraction to the skeleton

$$X^{\mathrm{an}} \to \Gamma$$

taking a connected component of $X^{\mathrm{an}} \smallsetminus \Gamma$ to its boundary point. Restricting to
$X(K)$ and extending linearly gives the tropicalization map on divisors

$$\mathrm{Trop} : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma).$$

This map respects rational equivalence of divisors, as follows.

Let $f \in K(X)$ be a rational function. We write $\mathrm{trop}(f)$ for the real valued
function on the skeleton Γ given by $y \mapsto \mathrm{val}_y(f)$. The function $\mathrm{trop}(f)$ is piecewise
linear with integer slopes. Furthermore, if y is a type-2 point and $\mathrm{trop}(f)(y) =
0$, then the residue $\overline{f}_y$ is a nonzero rational function on X_y whose slope along
an edge incident to y is the order of vanishing of $\overline{f}_y$ at the corresponding node.
This is the nonarchimedean Poincaré-Lelong Formula, due to Thuiller; see [Thu05]
and [BPR11, §5]. One immediate consequence of this formula is that the tropical
specialization map for rational functions

$$\mathrm{trop} : K(X)^* \to \mathrm{PL}(\Gamma)$$

is compatible with passing to principal divisors. More precisely, for any nonzero
rational function $f \in K(X)$, we have

$$\mathrm{Trop}(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f)).$$

Therefore, the tropicalization map on divisors respects equivalences and descends
to a natural map on Picard groups

$$\mathrm{Trop} : \mathrm{Pic}(X) \to \mathrm{Pic}(\Gamma).$$

Furthermore, since tropicalizations of effective divisors are effective, if D is a divisor
on X and f is a rational function in $\mathcal{L}(D)$, then $\mathrm{trop}(f)$ is in $R(\mathrm{Trop}(D))$. This
leads to the following version of Baker's Specialization Lemma.

Lemma 2.2. *Let D be a divisor on X. Then $r(\mathrm{Trop}(D)) \geq r(D)$.*

Here, the rank $r(D)$ is the dimension of the complete linear series of D on X.

Remark 2.3. Note that $\mathrm{trop}(\mathcal{L}(D))$ is often much smaller than $R(\mathrm{Trop}(D))$. It is difficult in general to determine which piecewise linear functions in $R(\mathrm{Trop}(D))$ are tropicalizations of rational functions in $\mathcal{L}(D)$.

3. Tropical Multiplication Maps

We now introduce a basic tropical lemma for studying linear dependence of rational functions and ranks of multiplication maps on linear series.

3.1. **Tropical independence.** Let $f_0, \ldots, f_r$ be rational functions on X. Suppose $\{f_0, \ldots, f_r\}$ is linearly dependent, so there are constants $c_0, \ldots, c_r$ in K, not all zero, such that

$$c_0 f_0 + \cdots + c_r f_r = 0.$$

Then, for any point $v \in X^{\mathrm{an}}$, the minimum of the valuations

$$\{\mathrm{val}_v(c_0 f_0), \ldots, \mathrm{val}_v(c_r f_r)\}$$

must occur at least twice. In particular, if $f_0, \ldots, f_r$ are linearly dependent in $K(X)$ then there are real numbers $b_0, \ldots, b_r$ such that the minimum of the piecewise linear functions $\{\mathrm{trop}(f_0) + b_0, \ldots, \mathrm{trop}(f_r) + b_r\}$ occurs at least twice at every point of the skeleton Γ. Here, take $b_j = \mathrm{val}(c_j)$ if c_j is nonzero, and otherwise make b_j sufficiently large such that $\psi_j + b_j$ is never minimal.

Definition 3.1. *A set of piecewise linear functions $\{\psi_0, \ldots, \psi_r\}$ is* tropically dependent *if there are real numbers $b_0, \ldots, b_r$ such that the minimum*

$$\min\{\psi_0(v) + b_0, \ldots, \psi_r(v) + b_r\}$$

occurs at least twice at every point v in Γ.

If there are no such real numbers $b_0, \ldots, b_r$ then we say $\{\psi_0, \ldots, \psi_r\}$ is *tropically independent.*

Lemma 3.2. *Let D and E be divisors on X, with $\{f_0, \ldots, f_r\}$ and $\{g_0, \ldots, g_s\}$ bases for $\mathcal{L}(D)$ and $\mathcal{L}(E)$, respectively. If $\{\mathrm{trop}(f_i) + \mathrm{trop}(g_j)\}_{ij}$ is tropically independent then the multiplication map*

$$\mu : \mathcal{L}(D) \otimes \mathcal{L}(E) \to \mathcal{L}(D + E)$$

is injective.

Proof. The elementary tensors $f_i \otimes g_j$ form a basis for $\mathcal{L}(D) \otimes \mathcal{L}(E)$. The image of $f_i \otimes g_j$ under μ is the rational function $f_i g_j$, and these are linearly independent, since their tropicalizations are tropically independent. $\square$

Remark 3.3. The main difficulty in applying this lemma is that one must prove the existence of rational functions in the algebraic linear series whose tropicalizations have the appropriate independence property. Finding such piecewise linear functions in the tropical linear series is not enough.

3.2. **Shapes of equivalent divisors.** Here we prove a technical proposition about how the tropical module structure on $R(D)$ is reflected in the shapes of divisors in $|D|$. The proposition will be particularly useful when combined with our notion of tropical dependence of piecewise linear functions.

Lemma 3.4. *Let D be a divisor on a metric graph Γ, with $\psi_0, \ldots, \psi_r$ piecewise linear functions in $R(D)$, and let*

$$\theta = \min\{\psi_0, \ldots, \psi_r\}.$$

Let $\Gamma_j \subset \Gamma$ be the closed set where $\theta = \psi_j$. Then $\operatorname{div}(\theta) + D$ contains a point $v \in \Gamma_j$ if and only if v is in either

(1) *the divisor $\operatorname{div}(\psi_j) + D$, or*
(2) *the boundary of Γ_j.*

Proof. If ψ_j agrees with θ on some open neighborhood of v, then $\operatorname{ord}_v(\theta) = \operatorname{ord}_v(\psi_j)$, and hence $\operatorname{div}(\theta) + D$ contains v if and only if $\operatorname{div}(\psi_j) + D$ does. On the other hand, if v is in the boundary of Γ_j then there is an edge containing v such that the incoming slope of θ along this edge is strictly greater than that of f_j, and the incoming slope of θ along any other edge containing v must be at least as large as that of ψ_j. By summing over all edges containing v we find that $\operatorname{ord}_v(\theta)$ is strictly greater than $\operatorname{ord}_v(\psi_j)$. Since $\operatorname{div}\psi_j + D$ is effective, by hypothesis, it follows that the coefficient of v in $\operatorname{div}(\theta) + D$ is strictly positive, as required. $\qquad\square$

Proposition 3.5. *Let D be a divisor on a metric graph Γ, with $\psi_0, \ldots, \psi_r$ in $R(D)$ and*

$$\theta = \min\{\psi_0, \ldots, \psi_r\}.$$

Let $\Gamma' \subset \Gamma$ be a connected subset, and suppose that $\operatorname{div}(\psi_j) + D$ contains a point in Γ' for all j. Then $\operatorname{div}(\theta) + D$ also contains a point in Γ'.

Proof. Let

$$\Gamma'_j = \{v \in \Gamma' \mid \psi_j(v) = \theta(v)\}.$$

Note that Γ'_j is a closed subset of Γ' for all j, and these subsets cover Γ'. Choose j such that Γ'_j is nonempty. If Γ_j is properly contained in Γ' then its boundary is nonempty, since Γ' is connected, and each of its boundary points is contained in $\operatorname{div}(\theta) + D$, by Lemma 3.4. On the other hand, if $\Gamma'_j = \Gamma'$, then $\operatorname{div}\theta + D$ is equal to $\operatorname{div}\psi_j + D$, and hence contains a point in Γ'. $\qquad\square$

4. THE CHAIN OF LOOPS WITH BRIDGES

We now restrict attention to the specific graph Γ shown in Figure 1, consisting of a chain of g loops separated by bridges. Throughout, we assume that the loops of Γ have generic edge lengths in the same sense as in [CDPR12], meaning that ℓ_i/m_i is never equal to the ratio of two positive integers whose sum is less than or equal to $2g - 2$.

4.1. **Reduced divisors.** Fix a point $v \in \Gamma$. Recall that an effective divisor D is v-reduced if the multiset of distances from v to points in D is lexicographically minimal among all effective divisors equivalent to D. Every effective divisor is equivalent to a unique v-reduced divisor, and the rank of a v-reduced divisor is bounded above by the coefficient of v. In particular, if D is a v-reduced divisor that does not contain v, then $r(D)$ is zero. See [Luo11, Proposition 2.1].

It is relatively straightforward to classify v-reduced divisors on Γ. We will only need the special case of w_g-reduced divisors. For each i, let γ_i be the ith loop minus

w_i, the union of the two half-open edges $[v_i, w_i)$, and let br_i be the half-open bridge $[w_i, v_{i+1})$. Note that Γ decomposes as a disjoint union

$$\Gamma = \gamma_1 \sqcup \mathrm{br}_1 \sqcup \cdots \sqcup \gamma_g \sqcup \{w_g\},$$
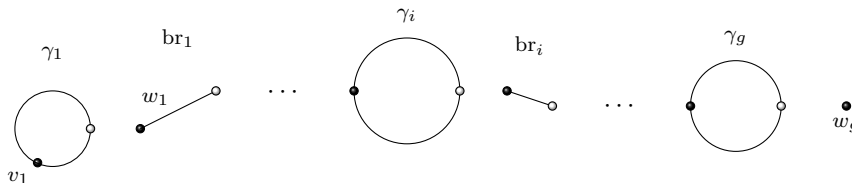
as shown.

FIGURE 2. A decomposition of Γ.

Proposition 4.1. *An effective divisor D is w_g-reduced if and only if it contains*

(1) *no points in the bridges $\mathrm{br}_1, \ldots, \mathrm{br}_{g-1}$, and*
(2) *at most one point in each cell $\gamma_1, \ldots, \gamma_g$.*

Proof. This is a straightforward application of Dhar's burning algorithm, as in [CDPR12, Example 2.6]. □

4.2. **The shape of a canonical divisor.** As mentioned in the introduction, our strategy is a proof by contradiction; we assume that a multiplication map has nonzero kernel and use Proposition 3.5 to construct a canonical divisor of *impossible shape*.

The following basic lemma restricts the possibilities for the shape of a canonical divisor on an arbitrary graph.

Lemma 4.2. *Let Γ' be a metric graph without loops, let $e_1, \ldots, e_g$ be open edges of Γ' whose complement is a spanning tree, and let D be an effective divisor equivalent to $K_{\Gamma'}$. Then at least one of the open edges $e_1, \ldots, e_g$ contains no point of D.*

Proof. Suppose each open edge $e_1, \ldots, e_g$ contains a point of D, let p_i be a point in e_i, and let $D' = p_1 + \cdots + p_g$. Since $K_{\Gamma'} - D'$ is effective, by construction, the tropical Riemann-Roch Theorem says that $r(D')$ is at least 1. However, Dhar's burning algorithm [Dha90] shows that D' is v-reduced for any point $v \in \Gamma'$. Since D' does not contain v, it follows that $r(D')$ is zero. □

On the chain of loops with bridges, we can use the classification of w_g-reduced divisors to refine this lemma as follows.

Lemma 4.3. *Let D be an effective divisor equivalent to K_Γ. Then D contains no point in at least one of the cells $\gamma_1, \ldots, \gamma_g$.*

Proof. Suppose each cell $\gamma_1, \ldots, \gamma_g$ contains a point of D. Let p_i be a point of D in γ_i, and let $D' = p_1 + \cdots + p_g$. Then $K_\Gamma - D'$ is equivalent to an effective divisor, by construction, so the tropical Riemann-Roch Theorem says that $r(D')$ is at least 1. However, D' is w_g-reduced by Proposition 4.1 and does not contain w_g, so $r(D')$ is zero. □

Remark 4.4. Note that the point p_i in the proof of Lemma 4.3 may be equal to v_i for some $2 \leq i \leq g$ in which case the complement of $\{p_1, \ldots, p_g\}$ does not contain a spanning tree. For this reason, the lemma does not follow from Lemma 4.2.

5. Preliminaries for the proof of injectivity

Let X be a curve over K with skeleton Γ, and let D_X be a divisor of degree d and rank r on X. To prove that X is Gieseker-Petri general we must show that the multiplication map

$$\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(K_X)$$

is injective. Given Lemma 3.2, a natural strategy is to show that there are bases $\{f_i\}$ and $\{g_j\}$ for $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$, respectively, such that the set of piecewise linear functions

$$\{\mathrm{trop}(f_i) + \mathrm{trop}(g_j)\}_{ij}$$

is tropically independent. We can prove the existence of such a basis exactly when the Brill-Noether number $\rho(g, r, d)$ is zero. The following section, which treats this special case, is not logically necessary for the proof of Theorem 1.1. However, the basic strategy that we use is the same as in the general case, only the details are simpler.

6. A special case: Brill-Noether number zero

The results of this sections are not used in the proof of Theorem 1.1, but working through this special case where $\rho(g, r, d)$ is zero before proceeding to the proof of the general case should be helpful for most readers. An overview of the argument is as follows.

We start by assuming that the multiplication map has a kernel, and therefore the tropicalization of the image under μ of any basis for $\mathcal{L}(D_X) \otimes \mathcal{L}(K_X - D_X)$ is tropically dependent. We use this tropical dependence together with Proposition 3.5 to construct a divisor in $|K_\Gamma|$ that violates Lemma 4.3, i.e. a canonical divisor of impossible shape. When the Brill-Noether number is zero, the bases for $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$ are explicit and canonically determined, and we only need to choose one basis for each. Additional subtleties in the general case include the choice of g different bases for $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$, one for each loop in Γ, and the application of Poincaré-Lelong to control the slopes of tropicalizations along the bridges. Furthermore, the bases are not explicit in the general case, but Lemma 7.2 gives the existence of bases with the required properties.

Suppose D_X is a divisor of degree d and rank r on X, with $\rho(g, r, d) = 0$, and let D be the v_1-reduced divisor equivalent to $\mathrm{trop}(D_X)$. There are only finitely many v_1-reduced divisors of degree d and rank r on Γ, and they are explicitly classified in [CDPR12]. These divisors correspond naturally and bijectively to the standard tableau with $(g - d + r)$ rows and $(r + 1)$ columns. Note that, since $\rho(g, r, d) = 0$, the genus g factors as

$$g = (r + 1)(g - d + r).$$

In particular, the entries in the tableau corresponding to D are the integers $1, \ldots, g$.

Fix the tableau corresponding to D. We label the columns from 0 to r, and the rows from 0 to $g - d + r - 1$. The tableau determines a Dyck path, consisting of a series of points $p_0, \ldots, p_g$ in $\mathbb{Z}^r$, as follows. We write $e_0, \ldots, e_{r-1}$ for the standard basis vectors on $\mathbb{Z}^r$. The starting and ending point of the Dyck path is

$$p_0 = p_g = (r, \ldots, 1),$$

and the ith step $p_i - p_{i-1}$ is equal to

- the standard basis vector e_j if i appears in the jth column of the tableau, for $0 \le j < r$, or

- the vector $(-1, \ldots, -1)$ if i appears in the last column.

The tableau properties exactly ensure that each p_i lies in the open Weyl chamber $x_0 > \cdots > x_{r-1} > 0$. We write $p_i(j)$ for the jth coordinate of p_i.

The divisor D can be recovered from the Dyck path as follows. The coefficient of v_1 is r. If i appears in the jth column of the tableau, for $1 \leq j < r$, then D contains the point on the ith loop at distance $p_i(j)$ modulo $(\ell_i + m_i)$ counterclockwise from v_i with coefficient 1. If i appears in the last column of the tableau, then D contains no point in the ith loop.

Remark 6.1. In this bijection, adjunction of divisors corresponds to transposition of tableaux [AMSW13, Theorem 39]. Therefore, the v_1-reduced divisor E equivalent to $\operatorname{trop}(K_X - D_X)$ is exactly the divisor corresponding to the transpose of the tableau for D.

Proposition 6.2. *For each integer $0 \leq j \leq r$, there is a unique divisor D_j equivalent to D such that $D_j - jv_1 - (r-j)w_g$ is effective. Moreover, γ_i contains no point of D_j if and only if i appears in the jth column of the tableau corresponding to D.*

Proof. The divisor D_r is exactly D. The remaining divisors D_j are constructed in the proof of [CDPR12, Proposition 4.10], by an explicit chip-firing procedure. One takes a pile of $r-j$ chips from v_1 and moves it to the right. The pile of chips changes size as it moves, and has $p_i(j)$ chips when it reaches v_i. As the pile moves across the ith loop, there is a single chip left behind in the interior of one of the edges unless i appears in the jth column, in which case the ith loop is left empty. When the pile reaches w_g, it has $p_g(j) = r - j$ chips. Since j chips were left at v_1 at the start of the procedure, $D_j - jv_1 - (r-j)w_g$ is effective. To see that D_j is the unique divisor equivalent to D with this property, note that $D_j - jv_1 - (r-j)w_g$ does not move; it is effective and contains no points on the bridges or at the vertices, and hence is v-reduced for every v in Γ. $\square$

Similarly, for $0 \leq k \leq g - d + r - 1$ there is a unique divisor E_k equivalent to the v_1-reduced adjoint divisor E such that $E_k - kv_1 - (g - d + r - 1 - k)w_g$ is effective, and γ_i contains no point of E_k if and only if i appears in the kth row of the tableau.

It follows that the g divisors $D_j + E_k$ are distinct and correspond to the loops of Γ, as follows.

Corollary 6.3. *The connected subset $\gamma_i \subset \Gamma$ contains no point of $D_j + E_k$ if and only if i appears in the jth column and kth row of the tableau corresponding to D.*

Proposition 6.4. *There is a basis $f_0, \ldots, f_r$ for $\mathcal{L}(D_X)$ such that*

$$\operatorname{Trop}(D_X + \operatorname{div}(f_j)) = D_j.$$

Proof. Let x and y be points in $X(K)$ specializing to v_1 and w_g, respectively. Since D_X has rank r, there is a rational function $f_j \in \mathcal{L}(D_X)$ such that $D_X + \operatorname{div}(f_j)$ contains x with coefficient at least j and y with coefficient at least $r - j$. Then $\operatorname{Trop}(D_X + \operatorname{div}(f_j))$ is an effective divisor and contains v_1 and w_g with coefficient at least j and $r - j$, respectively. By Proposition 6.2, $\operatorname{Trop}(D_X + \operatorname{div}(f_j))$ must be equal to D_j. $\square$

Similarly, there is a basis $\{g_0, \ldots, g_{g-d+r-1}\}$ for $\mathcal{L}(K_X - D_X)$ such that

$$\operatorname{Trop}(K_X - D_X + \operatorname{div}(g_k)) = E_k.$$

We proceed to study the piecewise linear functions

$$\phi_j = \operatorname{trop}(f_j) \quad \text{and} \quad \psi_k = \operatorname{trop}(g_k).$$

Note that $D + \operatorname{div}(\phi_j) = D_j$ and $E + \operatorname{div}(\psi_k) = E_k$. By Proposition 6.2, this determines each ϕ_j and ψ_k up to an additive constant, .

Theorem 6.5. *The set of g piecewise linear functions $\{\phi_j + \psi_k\}_{jk}$ is tropically independent.*

Proof. Suppose that $\{\phi_j + \psi_k\}_{jk}$ is tropically dependent. Then there exist real numbers b_{jk} such that the minimum

$$\theta = \min_{j,k}\{\phi_j + \psi_k + b_{jk}\}$$

occurs at least twice at every point in Γ. Note that $D + E + \operatorname{div}(\theta)$ is an effective canonical divisor, since $R(D + E)$ is a tropical module and D and E are adjoint.

We claim that $D + E + \operatorname{div}\theta$ contains a point in each γ_i. Choose j_0 and k_0 such that i appears in the j_0th row and k_0th column of the tableau corresponding to D. Then Corollary 6.3 says that $D + E + \operatorname{div}(\phi_j + \psi_k + b_{jk})$ contains a point in γ_i for $(j, k) \neq (j_0, k_0)$. Also, since the minimum of $\{\phi_j + \psi_k + +b_{jk}\}$ occurs at least twice at every point of Γ, we have

$$\theta = \min_{(j,k)\neq(j_0,k_0)}\{\phi_j + \psi_k + b_{jk}\}.$$

Therefore, by Proposition 3.5, the divisor $D + E + \operatorname{div}(\theta)$ contains a point in γ_i, as claimed.

We have shown that $D + E + \operatorname{div}(\theta)$ is an effective canonical divisor that contains a point in each of $\gamma_1, \ldots, \gamma_g$. But this is impossible, by Lemma 4.3. $\square$

7. Proof of Theorem 1.1

As in the previous two sections, let X be a smooth projective curve of genus g over K with skeleton Γ. Since skeletons are invariant under base change with respect to extensions of algebraically closed valued fields, we can and do assume that K is spherically complete.

Remark 7.1. Spherical completeness is equivalent to completeness for discretely valued fields, but stronger in general. We use spherical completeness only in the proof of Lemma 7.2, to ensure that normed K-vector spaces have orthogonal bases.

Let D_X be an effective divisor on X. We must show that the multiplication map

$$\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(D)$$

is injective. This is trivial if $\mathcal{L}(K_X - D_X)$ is zero, so we assume there is an effective divisor E_X equivalent to $K_X - D_X$. We may also assume v_1 and w_g are type-2 points and choose type-2 points w_0 and v_{g+1} in the connected components of $X^{\mathrm{an}} \smallsetminus \Gamma$ with boundary points v_1 and w_g, respectively. Then

$$V = \{v_1, \ldots, v_{g+1}, w_0, \ldots, w_g\}$$

is a semistable vertex set, with skeleton $\Gamma_V \supset \Gamma$ as shown.

FIGURE 3. The skeleton Γ_V.

Let $\mathcal{X}_V$ be the semistable model of X associated to V, with X_i the component of the special fiber $\overline{\mathcal{X}}_V$ corresponding to v_i, and $x_i \in X_i$ the node corresponding to the edge $e_i = [w_{i-1}, v_i]$, for $1 \le i \le g$.

Recall that the reduction of f in $\kappa(X_i)^*$ is the residue of af with respect to the valuation val_{v_i} on $K(X)$, where $a \in K^*$ is chosen such that $\mathrm{val}_{v_i}(af) = 0$ [AB12] . This reduction is defined only up to multiplication by elements of κ^*, but its order of vanishing at x_i is independent of all choices. Similarly, if $f_0, \ldots, f_r$ are rational functions in $K(X)^*$, then the κ-span of their reductions in $\kappa(X_i)$ is independent of all choices. In particular, it makes sense to talk about whether these reductions are linearly independent.

Lemma 7.2. *Let D be a divisor of rank r on X. For each $1 \le i \le g$, there is a basis $f_0, \ldots, f_r$ for $\mathcal{L}(D)$ such that*

(1) *the reductions of $f_0, \ldots, f_r$ in $\kappa(X_i)$ have distinct orders of vanishing at x_i, and*

(2) *the reductions of $f_0, \ldots, f_r$ in $\kappa(X_{i+1})$ are linearly independent.*

Proof. We consider $\mathcal{L}(D_X)$ as a normed vector space over K, with respect to the norms $|\ |_i$ and $|\ |_{i+1}$ whose logarithms are $-\mathrm{val}(v_i)$ and $-\mathrm{val}(v_{i+1})$, respectively, and use the basic properties of nonarchimedean normed vector spaces developed in [BGR84, Chapter 2]. Since K is spherically complete, the vector space $\mathcal{L}(D_X)$ is K-cartesian [BGR84, 2.4.4.2], and since v_i and v_{i+1} are type-2 points, the image of $\mathcal{L}(D_X)$ under each of these norms is equal to the image of K under its given norm. Therefore, $\mathcal{L}(D_X)$ is strictly K-cartesian [BGR84, 2.5.1.2], which means that all of its subspaces have orthonormal bases. So, first choose an orthonormal basis for $\mathcal{L}(D_X)$ with respect to $|\ |_i$. The reductions of these basis elements are linearly independent [BGR84, 2.5.1.3], so we can take suitable combinations with coefficients in R^* to ensure that they have distinct orders of vanishing at x_i.

Let $f_0, \ldots, f_r$ be a basis for $\mathcal{L}(D_X)$ whose reductions in $\kappa(X_i)$ have strictly decreasing order of vanishing at x_i. Then, for each j, we can replace f_j by a suitable linear combination of $f_0, \ldots, f_j$ that is orthogonal to the span of $f_0, \ldots, f_{j-1}$ with respect to $|\ |_{i+1}$. This does not change the order of vanishing at x_i of the reduction in $\kappa(X_i)$, but ensures that the reductions in $\kappa(X_{i+1})$ are linearly independent. $\square$

This lemma, closely analogous to [EH83, Lemma 1.2], will be especially useful in combination with the following identity relating orders of vanishing of reductions of rational functions to the slopes of their tropicalizations. For any piecewise linear function ψ on Γ_V, we write $s_i(\psi)$ for the incoming slope of ψ at v_i along e_i. Suppose $\psi = \mathrm{trop}(f)$ for some rational function f in $K(X)^*$. Then Thuillier's nonarchimedean analytic Poincaré-Lelong formula [Thu05, BPR11] says that $s_i(\mathrm{trop}(f))$ is the order of vanishing at x_i of the reduction of f in $\kappa(X_i)$.

Fix a basis $f_0, \ldots, f_r$ for $\mathcal{L}(D_X)$ whose reductions in $\kappa(X_i)$ have distinct orders of vanishing at x_i, and whose reductions at X_{i+1} are linearly independent. Let $a_0, \ldots, a_r$ be constants in K. Define

$$\psi = \mathrm{trop}(a_0 f_0 + \cdots + a_r f_r).$$

and

$$\psi' = \min\{\mathrm{trop}(f_0) + \mathrm{val}(a_0), \ldots, \mathrm{trop}(f_r) + \mathrm{val}(a_r)\}.$$

Note that

$$\psi(v) \ge \psi'(v)$$

for all v, with equality when v is equal to v_i or v_{i+1}. This is because the reductions of the $a_j f_j$ in both $\kappa(X_i)$ and $\kappa(X_{i+1})$ are linearly independent.

Proposition 7.3. *The piecewise linear functions ψ and ψ' are equal on some nonempty interval $(v, v_i) \subset e_i$.*

Proof. The two functions ψ and ψ' agree at any point v where the minimum of $\{\mathrm{trop}(f_0)(v) + \mathrm{val}(a_0), \ldots, \mathrm{trop}(f_r)(v) + \mathrm{val}(a_r)\}$ occurs only once. By construction, the reductions of $f_0, \ldots, f_r$ in $\kappa(X_i)$ have distinct orders of vanishing at x_i, so the Poincaré-Lelong formula says that $\mathrm{trop}(f_0), \ldots, \mathrm{trop}(f_r)$ have distinct incoming slopes at v_i along e_i. It follows that the minimum occurs only once on some open interval (v, v_i), and ψ and ψ' agree on this interval. $\square$

The final ingredient in our proof of Theorem 1.1 is the following proposition relating slopes along bridges to shapes of divisors in a linear series on Γ_V.

Proposition 7.4. *Let D be an effective divisor of degree at most $2g - 2$ on Γ_V, and let $\psi_0, \ldots \psi_r \in R(D)$ be piecewise linear functions with distinct incoming slopes at v_i, for some $1 \le i \le g$. Then at most one of the divisors $D + \mathrm{div}(\psi_0), \ldots, D + \mathrm{div}(\psi_r)$ contains no point in γ_i.*

Proof. Let Γ' be the union of the ith loop together with a small closed subsegment of $[v, v_i] \subset [w_{i-1}, v_i]$ along which $\psi_0, \ldots, \psi_r$ all have constant slope. We may choose v sufficiently close to v_i so that D contains no points in $[v, v_i)$. Let $D' = D|_{\Gamma'}$ and $\psi'_j = \psi_j|_{\Gamma'}$. Note that the coefficient of v in $\mathrm{div}(\psi'_j)$ is $-s_i(\psi_j)$, and $D' + \mathrm{div}(\psi'_j)$ agrees with $D + \mathrm{div}(\psi_j)$ on γ_i. We now show that at most one of the divisors $D' + \mathrm{div}(\psi'_j)$ contains no point in γ_i.

Suppose $D' + \mathrm{div}(\psi'_j)$ and $D' + \mathrm{div}(\psi'_k)$ both contain no point in γ_i. Then both of these divisors are supported at v and w_i. Subtracting one from the other, we find an equivalence of divisors

$$(s_i(\psi_j) - s_i(\psi_k))\, v \ \sim \ (s_i(\psi_j) - s_i(\psi_k))\, w_i$$

on Γ'. Note that $s_i(\psi_j)$ is bounded above by the sum of the coefficients of D at points to the left of v_i and bounded below by minus the sum of its coefficients at v_i and to the right. Similarly, $-s_i(\psi_k)$ is bounded above by the sum of the coefficients of D at v_i and to the right, and bounded below by minus the sum of its coefficients at points to the left of v_i. Therefore, $|s_i(\psi_j) - s_i(\psi_k)|$ is bounded by the degree of D. The equivalence above then implies that ℓ_i/m_i is a ratio of two positive integers whose sum is less than or equal to the degree of D, contradicting the genericity hypothesis on the edge lengths. $\square$

Proof of Theorem 1.1. Suppose the multiplication map

$$\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(E_X) \to \mathcal{L}(K_X)$$

has nonzero kernel. For $1 \le i \le g$, let $\{f_0^i, \ldots, f_r^i\}$ be a basis for $\mathcal{L}(D_X)$ consisting of rational functions whose reductions in $\kappa(X_i)$ have distinct orders of vanishing at x_i and whose reductions in $\kappa(X_{i+1})$ are linearly independent. Similarly, let $\{g_0^i, \ldots, g_{g-d+r-1}^i\}$ be a basis for $\mathcal{L}(E_X)$ consisting of rational functions whose reductions in $\kappa(X_i)$ have distinct orders of vanishing at x_i and whose reductions in $\kappa(X_{i+1})$ are linearly independent.

Fix an element in the kernel of μ. Then, for each i, we can express this element uniquely as a sum of elementary tensors

$$\sum_{j,k} a_{j,k}^i f_j^i \otimes g_k^i.$$

Define a piecewise linear function

$$\theta_i = \min_{j,k}\{\mathrm{trop}(f_j^i) + \mathrm{trop}(g_k^i) + \mathrm{val}(a_{j,k}^i)\},$$

and note that the minimum must occur at least twice at every point in Γ_V.

Replacing $\{f_0^i, \ldots, f_r^i\}$ by $\{af_0^i, \ldots, af_r^i\}$ for some $a \in K^*$, we may assume that $\theta_i(v_{i+1}) = \theta_{i+1}(v_{i+1})$ for $1 \leq i < g$. Then there is a continuous piecewise linear function θ on Γ_V that agrees with θ_i between v_i and v_{i+1} for $1 \leq i < g$, and agrees with θ_g on γ_g. Let

$$D = \mathrm{Trop}(D_X) \quad \text{and} \quad E = \mathrm{Trop}(E_X).$$

We claim that $D + E + \mathrm{div}(\theta)$ is effective and contains a point in γ_i, for $1 \leq i \leq g$.

First we show that $D + E + \mathrm{div}(\theta)$ is effective. In the open subgraph between v_i and v_{i+1}, the divisor $D + E + \mathrm{div}(\theta)$ agrees with $D + E + \mathrm{div}(\theta_i)$, which is effective because $R(D + E)$ is a tropical module that contains $\mathrm{trop}(f_j^i) + \mathrm{trop}(g_k^i)$ for all j and k. It remains to check that the coefficient of v_i is nonnegative. Since $D + E + \mathrm{div}(\theta_i)$ is effective, it will suffice to show

$$s_i(\theta_{i-1}) \geq s_i(\theta_i).$$

We prove this by changing coordinates in two steps, first replacing the basis $\{f_j^i\}_j$ with $\{f_j^{i-1}\}_j$ and then replacing the basis $\{g_k^i\}_k$ with $\{g_k^{i-1}\}_k$.

Fix k, write

$$\sum_j a_{j,k}^i f_j^i = \sum_j b_{j,k} f_j^{i-1},$$

and define

$$\theta' = \min_{j,k}\{\mathrm{trop}(f_j^{i-1}) + \mathrm{trop}(g_j^i) + \mathrm{val}(b_{j,k})\}.$$

Note that

$$\min_j\{\mathrm{trop}(f_j^{i-1})(v_i) + \mathrm{val}(b_{j,k})\} = \min_j\{\mathrm{trop}(f_j^i)(v_i) + \mathrm{val}(a_{j,k}^i)\},$$

since the reductions of both $\{f_j^i\}_j$ and $\{f_j^{i-1}\}_j$ in $\kappa(X_i)$ are linearly independent. By adding the constant $g_k^i(v_i)$ and taking the minimum over all k, we see that

$$\theta'(v_i) = \theta(v_i).$$

We now examine the slopes $s_i(\theta)$ and $s_i(\theta')$. At any point v on the edge between w_{i-1} and v_i, we have

$$\mathrm{trop}\Big(\sum_j b_{j,k} f_j^{i-1}\Big)(v) \geq \min_j\big\{\mathrm{trop}(b_{j,k}) + \mathrm{trop}(f_j^{i-1})\big\}(v).$$

Since this inequality holds with equality at v_i, it follows that

$$s_i\big(\mathrm{trop}\big(\sum_j b_{j,k} f_j^{i-1}\big)\big) \leq s_i\big(\min_j\big\{\mathrm{trop}(b_{j,k}) + \mathrm{trop}(f_j^{i-1})\big\}\big).$$

Now Proposition 7.3 tells us that, on some nonempty interval $(v, v_i) \subset e_i$,

$$\mathrm{trop}\Big(\sum_j b_{j,k} f_j^{i-1}\Big) = \min_j\big\{\mathrm{trop}(a_{j,k}^i) + \mathrm{trop}(f_j^i)\big\}.$$

Taking the minimum over those values of k for which $\min_j\{\mathrm{trop}(a_{j,k}^i) + \mathrm{trop}(f_j^i)\}(v_i) + \mathrm{trop}(g_k^i)(v_i) = \theta(v_i)$, we see that

$$s_i(\theta_i) \leq s_i(\theta').$$

A similar argument, fixing j and replacing the basis $\{g_k^i\}$ with g_k^{i-1} shows that $s_i(\theta') \leq s_i(\theta_{i-1})$, as required. This proves that $D + E + \mathrm{div}(\theta)$ is effective. It remains to show that $D + E + \mathrm{div}(\theta)$ contains a point in each cell $\gamma_1, \ldots, \gamma_g$.

We now show that $D+E+\mathrm{div}(\theta)$ contains a point in γ_i. By Proposition 7.4, there is at most one pair of indices (j, k) such that $D+E+\mathrm{div}(\mathrm{trop}(f_j^i))+\mathrm{div}(\mathrm{trop}(g_k^i))+$

$\mathrm{val}(a_{j,k}^i)$ contains no point in γ_i. Call this pair (j_0, k_0) if it exists. By hypothesis, the minimum of these piecewise linear functions occurs at least twice at every point, so

$$\theta_i = \min_{(j,k) \neq (j_0,k_0)} \big\{ \, \mathrm{trop}(f_j^i)) + \mathrm{div}(\mathrm{trop}(g_k^i)) + \mathrm{val}(a_{j,k}^i) \big\}.$$

Therefore, by Proposition 3.5, $D + E + \mathrm{div}(\theta_i)$ contains a point in γ_i. Now, $D + E + \mathrm{div}(\theta)$ agrees with $D + E + \mathrm{div}(\theta_i)$ on $\gamma_i \smallsetminus \{v_i\}$. Furthermore, since $s_i(\theta_i) \leq s_i(\theta_{i-1})$, the coefficient of v_i in $D + E + \mathrm{div}(\theta)$ is greater than or equal to the coefficient of v_i in $D + E + \mathrm{div}(\theta_i)$. It follows that $D + E + \mathrm{div}(\theta)$ also contains a point in γ_i, as claimed.

Pushing forward the divisor $D + E + \mathrm{div}(\theta)$ under the natural contraction $\Gamma_V \to \Gamma$ gives an effective canonical divisor that contains a point in each cell $\gamma_1, \ldots, \gamma_g$. But this is impossible, by Lemma 4.3. $\square$

References

[AB12]   O. Amini and M. Baker. Linear series on metrized complexes of algebraic curves. preprint arXiv:1204.3508, 2012.

[ACGH85]   E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.

[ACP12]   D. Abramovich, L. Caporaso, and S. Payne. The tropicalization of the moduli space of curves. preprint arXiv:1212.0373, 2012.

[AMSW13]   R. Agrawal, G. Musiker, V. Sotirov, and F. Wei. Involutions on standard Young tableaux and divisors on metric graphs. *Electron. J. Combin.*, 20(3), 2013.

[Bak08]   M. Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.

[BGR84]   S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[BJM+12]   V. Baratham, D. Jensen, C. Mata, D. Nguyen, and S. Parekh. Toward a tropical proof of the Gieseker-Petri Theorem. To appear in Collectanea Math., 2012.

[BN07]   M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[BPR11]   M. Baker, S. Payne, and J. Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. preprint, arXiv:1104.0320v1, 2011.

[CDPR12]   F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2):759–776, 2012.

[CLMTiB12] A. Castorena, A. López Martín, and M. Teixidor i Bigas. Petri map for vector bundles near good bundles. preprint, arXiv:1203.0983, 2012.

[Dha90]   D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[EH83]   D. Eisenbud and J. Harris. A simpler proof of the Gieseker-Petri theorem on special divisors. *Invent. Math.*, 74(2):269–280, 1983.

[EH86]   D. Eisenbud and J. Harris. Limit linear series: basic theory. *Invent. Math.*, 85(2):337–371, 1986.

[GH80]   P. Griffiths and J. Harris. On the variety of special linear systems on a general algebraic curve. *Duke Math. J.*, 47(1):233–272, 1980.

[Gie82]   D. Gieseker. Stable curves and special divisors: Petri's conjecture. *Invent. Math.*, 66(2):251–275, 1982.

[GK08]   A. Gathmann and M. Kerber. A Riemann-Roch theorem in tropical geometry. *Math. Z.*, 259(1):217–230, 2008.

[HMY12]   C. Haase, G. Musiker, and J. Yu. Linear systems on tropical curves. *Math. Z.*, 270(3-4):1111–1140, 2012.

[Laz86]   R. Lazarsfeld. Brill-Noether-Petri without degenerations. *J. Differential Geom.*, 23(3):299–307, 1986.

[Len12]   Y. Len. First steps in Brill-Noether theory of tropical curves. preprint, arXiv:1209.6309v1, 2012.

[LPP12]   C.-M. Lim, S. Payne, and N. Potashnik. A note on Brill-Noether theory and rank-determining sets for metric graphs. *Int. Math. Res. Not. IMRN*, (23):5484–5504, 2012.

[Luo11]   Y. Luo. Rank-determining sets of metric graphs. *J. Combin. Theory Ser. A*, 118:1775–1793, 2011.

[MZ08]   G. Mikhalkin and I. Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008.

[Oss11]   B. Osserman. A simple characteristic-free proof of the Brill-Noether theorem. preprint, arXiv:1108.4967v1, 2011.

[Thu05]   A. Thuillier. *Théorie du potentiel sur ler courbes en géométrie analytique non archimédienne. Applications à la theéorie d'Arakelov.* PhD thesis, University of Rennes, 2005.