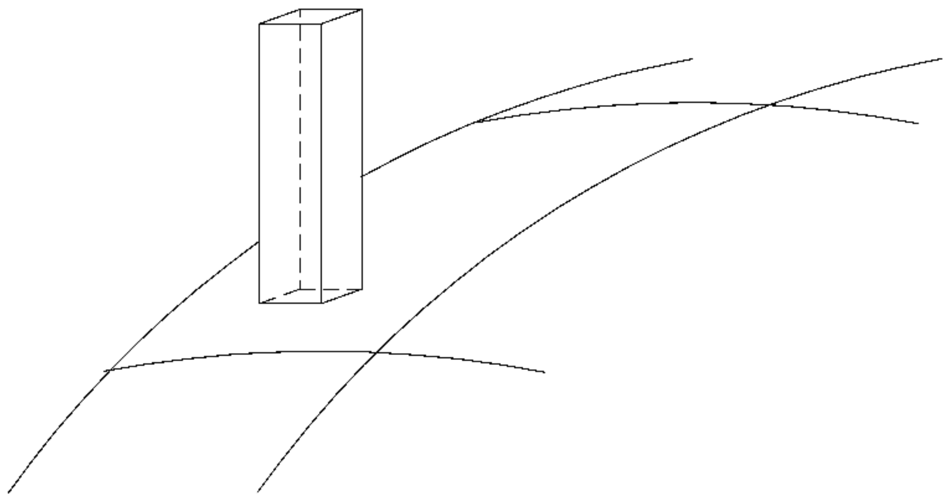
AP Calculus BC Project
Atmospheric Pressure

Atmospheric pressure on the Earth's surface is due to the weight of the atmosphere above that point. Let's consider a column of air over a one square inch section extending upward from the Earth's surface without stopping.



This column of air has weight. The weight of this column of air in pounds is numerically equal to the **atmospheric pressure** in pounds per square inch (psi) at the surface. To understand this we only need to know that pressure is force per unit area. We have already discussed the fact that weight is force. So at any height x above the surface of the Earth, the atmospheric pressure at that height is numerically equal to the weight of that part of the air column above height x. Thus, we can see why the air pressure decreases as our altitude increases – there is less air above us at higher altitudes.

We will construct two different mathematical models of atmospheric pressure. The first model is simplified since it does not take into account the variation of temperature with pressure. We use it though to prepare for the second model which will take temperature variation into account.

The Simplified Model

Let's consider a column of air as described above. Let

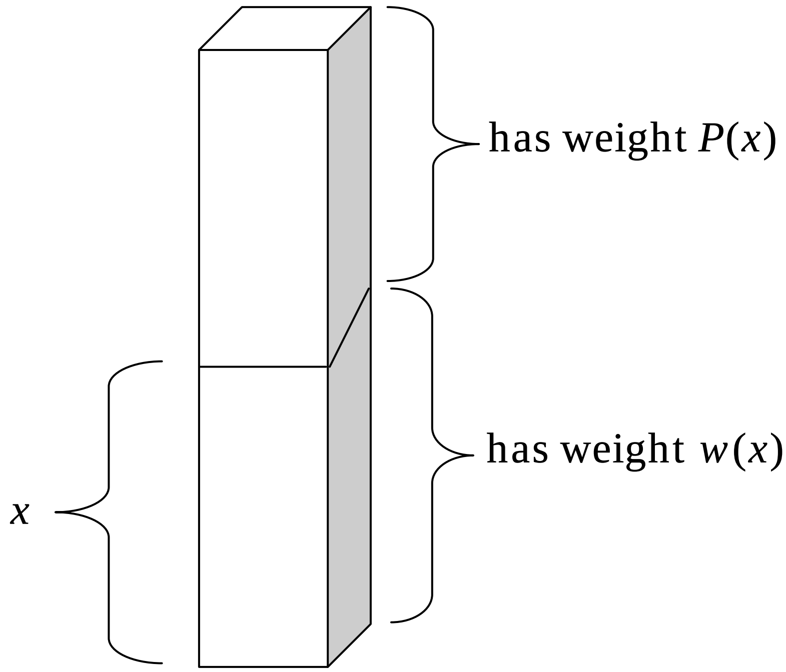
- P_0 denote the total weight of this column in pounds;
- $P(x)$ denote the atmospheric pressure in pounds per square inch at x inches above the surface of the Earth, for any $x \geq 0$; and
- $w(x)$ denote the weight, in pounds, of that portion of the column from the surface to the height of x inches above the surface, for any $x \geq 0$.

It should be clear from these definitions that the following equations hold:

$$P(0) = P_0$$

$$P(x) = P_0 - w(x) \qquad \text{for any } x \geq 0.$$

You should notice that as x increases the function $P(x)$ decreases at the same time that the function $w(x)$ increases. This should come as no surprise since their sum is a constant. To make the mathematics nicer, we will assume that these functions are each continuously differentiable.



If we were to weigh samples of air at various altitudes we would normally find that for a fixed volume, samples of air taken at low altitudes are heavier than samples taken at higher altitudes. For any $x \ge 0$ let $\rho(x)$ denote the weight of a cubic inch of air in our column at the height x inches above the surface of the Earth. This is called the **atmospheric density** at height x. Under normal conditions ρ is a decreasing function of x.

Put $\rho_0 = \rho(0)$, the atmospheric density at the surface of the column. Now, how fast does the weight of the air in the column change as we change the height? This is the difference quotient:

$$\frac{w(x+\Delta x)-w(x)}{\Delta x}.$$

Since the density is the weight of a cubic inch of air at height x, we see that this difference quotient is just the average density of the air in pounds per cubic inch in that part of the column. Since,

$$w'(x)=\lim_{\Delta x\to 0}\frac{w(x+\Delta x)-w(x)}{\Delta x}$$

and this represents the density of the atmosphere at height x, which is just $\rho(x)$, we have

$$w'(x)=\rho(x).$$

If we differentiate our first equation and apply this above equation we see that

$$P'(x)=-\rho(x) \quad \text{for any } x \ge 0.$$

Now we will make two simplifying assumptions to construct our model of the atmosphere:

1. The chemical composition of the atmosphere is uniform and independent of the height. This means that we are assuming that the ratios of the various gasses that make up the atmosphere are independent of the height.
2. The temperature of the atmosphere is independent of the height.

Now, we will apply these assumptions along with the assumption that the atmosphere is an ideal gas so we can apply Boyle's Law. This states that the density of a gas is proportional to its pressure. Thus, for any $x \geq 0$

$$\frac{P(x)}{P_0} = \frac{\rho(x)}{\rho_0}$$

or

$$\rho(x) = \frac{\rho_0}{P_0} P(x)$$

Combining this with the previous equation gives us the following differential equation:

$$\frac{dP}{dx}(x) = -\frac{\rho_0}{P_0} P(x)$$

We know how to solve this differential equation. You should solve the equation and you will have a solution that works for any $x \geq 0$. The solution relates the atmospheric pressure with height. For the sake of practicality we need to modify our solution so that x is measured in feet rather than inches. Thus, our solution should look like:

$$P(x) = P_0 \exp\left(\frac{-12\rho_0 x}{P_0}\right).$$

Given the assumptions of this section and given that $P_0 = 14.7$ psi and $\rho_0 = 4.34 \times 10^{-5}$ lbs/cu in, find the atmospheric pressure at 20,000 feet.

Problem 1: *Given the assumptions of this section and the values of P_0 and ρ_0 from above, at what height is the atmospheric pressure ½ of the pressure at the surface.*

Problem 2: *Given that $P_0 = 14.7$ psi and that the Earth's radius is 3,960 miles, what is the total weight of the Earth's atmosphere in* ***tons****. The surface area of a sphere is given by $S = 4\pi r^2$.*

A More Complicated Model

In this model we will delete the second assumption about the independence of temperature and altitude from the first model. We will measure temperature in degrees Kelvin, the absolute scale. The relationship between degrees Kelvin and degrees Celsius is

$$C = K - 273.1$$

In the Kelvin system 0° K is absolute zero — all motion even at the atomic level stops. Water freezes at 273.1° K and boils at 373.1° Kelvin. Using the standard change from

Celsius to Fahrenheit with the above equation, the relationships between Fahrenheit and Kelvin are given by:

$$F = 1.8K - 459.58$$

$$K = \frac{5}{9}(F + 459.58)$$

For any $x \geq 0$ let $T(x)$ denote the temperature in degrees Kelvin of the atmosphere at height x in inches above the surface of the planet. Let $T_0 = T(0)$ be the temperature at the surface. In the previous section we applied Boyle's Law of ideal gasses to get our differential equation. Since we are including the information about temperature in this model, we will need to impose another law on our atmosphere: Charles' Law of Ideal Gasses states that the gas density varies inversely with temperature when measured on an absolute scale, such as the Kelvin scale. (This is the reason for working in degrees Kelvin.) Thus, the density of the atmosphere is given by

$$\rho(x) = \frac{\rho_0 T_0 P(x)}{P_0 T(x)}.$$

Hence

$$\frac{P'(x)}{P(x)} = -\frac{\rho_0 T_0}{P_0 T(x)}$$

Thus, for any $x \geq 0$ we have

$$\int_0^x \frac{P'(u)}{P(u)} du = -\frac{\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du$$

$$\ln\left(\frac{P(x)}{P_0}\right) = -\frac{\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du$$

$$P(x) = P_0 e^{-\frac{\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du}$$

where x is the height in inches above the surface of the Earth. Adjusting the formula as in the previous section for measuring x in feet is just a change of variables and we get

$$P(x) = P_0 \exp\left(-\frac{12\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du\right)$$

where x is measured in feet and $T(x)$ is the temperature at a height of x feet above the surface of the Earth.

General aviation pilots use the rule:

> *The temperature of the atmosphere decreases linearly with height at a rate of 2° C (Celsius) per 1,000 feet of altitude.*

Since a difference of 2° Celsius is the same as a difference of 2° Kelvin, we can translate this rule into the formula:

$$T(x) = T_0 - .002x$$

where x is height measured in feet.

Problem 3: *Now we have enough to find the pressure. If* $P_0 = 14.7$ *psi,* $\rho_0 = 4.34 \times 10^{-5}$ *lbs/cu in and* $T_0 = 293^\circ K$*,*

1. *what is the atmospheric pressure at 20,000 feet above the surface of the Earth?*
2. *at what height is the atmospheric pressure half of the surface pressure?*

How do these answers compare with those of the first model?

An Application to Meteorology

We would normally expect that the density $\rho(x)$ of the atmosphere would decrease as x increases. Under these conditions we call the atmosphere *stable*. Otherwise, we call the atmosphere *unstable*. In an unstable atmosphere a given volume of air above would weigh at least as much as an equal volume of air below. Under these circumstances there would be a vertical motion of air causing winds and down drafts. We said that the atmosphere is unstable if ρ is non-decreasing, or if

$$\rho'(x) \geq 0 .$$

Under the assumptions of our second model, we know that

$$\rho(x) = \frac{\rho_0 T_0 P(x)}{P_0 T(x)}$$

and we have just found

$$P(x) = P_0 \exp\left(-\frac{12\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du \right)$$

where x is measured in feet and T is measured in degrees Kelvin. Since the atmosphere is unstable if $\rho'(x) \geq 0$.

$$\rho(x) = \frac{\rho_0 T_0}{T(x)} \exp\left(-\frac{12\rho_0 T_0}{P_0} \int_0^x \frac{1}{T(u)} du \right)$$

Differentiate ρ with respect to x, simplify it, and find where the derivative is 0.

Thus, the atmosphere is unstable if

$$T'(x) \leq \frac{-12\rho_0 T_0}{P_0}.$$

Assume that T drops linearly with height, $T_0 = 70^\circ$F, $\rho_0 = 4.34 \times 10^{-5}$ lbs/cu in, and $P_0 = 14.7$ psi. Find the maximum temperature at 1,000 feet so that the atmosphere is unstable. The assumption of linearity requires that

$$T(x) = T_0 - kx$$

for some positive constant k. Thus, by our inequality we have

$$-k \leq \frac{-12\rho_0 T_0}{P_0}$$
$$k \geq 0.0104$$

To achieve maximum temperature, we will require that k be as small as possible, k=0.0104, and so

$$T(x) = 294.2111 - 0.0104x.$$

Thus, for x = 1000, we have T(1000) = 283.8111° K or 51.28° F. Note that this gives a temperature drop of 10.4° Celsius per 1,000 feet, or 18.7° Fahrenheit per 1,000 feet.

Problem 4: *Assuming our second model, $P_0 = 14.7$, $\rho_0 = 4.34 \times 10^{-5}$, and $T_0 = 92\,°$ F, what is the atmospheric pressure at 10,000, 20,000, 30,000, 40,000, 50,000 feet? How about at 8 miles and 20 miles?*

Problem 5: *Assume the second model and the data in Problem 4. What is the maximum temperature at 36,000 feet so that the atmosphere could be unstable?*