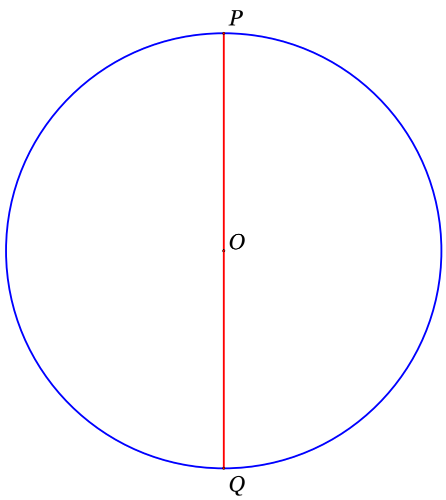
Pointwise Convergence and Length



Start with a circle of diameter d. The circumference is then πd and the distance along the circle from P to Q is

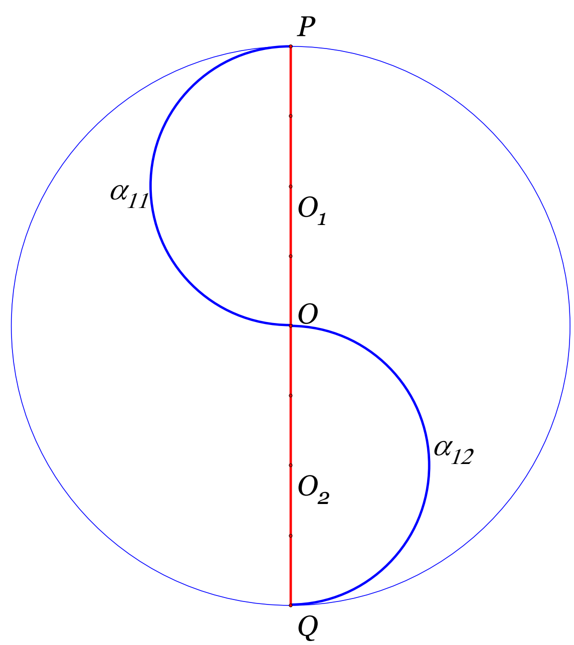
$$d_C = \pi \frac{d}{2} = \pi r \,.$$

The straight line distance from P to Q, through O, is $d_l = d = 2r$.

This is simple and straightforward. What we intend to do now is to find the midpoints of OP and OQ, call them O_1 and O_2. Letting $|PQ|$ denote the length of the segment PQ, then

$$|PO_1| = |O_1O| = |O_2O| = |O_2Q| = \frac{r}{2} = r_2 \,.$$

Construct the semicircles of radius r_2 centered at O_1 and O_2. The pattern shown below was chosen in order to show the two semicircles more clearly and because it gives a cooler diagram.



Now the distance from P to Q along the straight line has not changed. It is still $2r$.

How about the distance from P to Q by taking the semicircle of length α_{11} followed by that of length α_{12}? Well, the semicircle has length π × radius or

$$\alpha_{11} = \alpha_{12} = \pi r_2 = \frac{\pi r}{2} \,.$$

This means that the trip from P to Q along the two semicircles has length $d_{C2} = \pi r$, just as before!

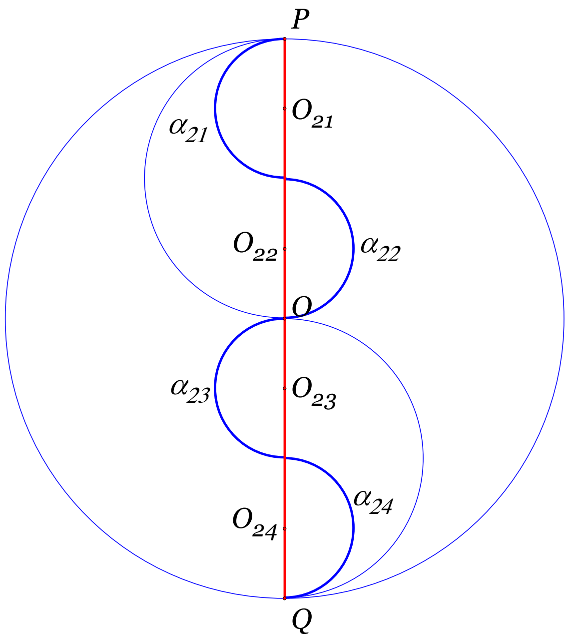
Okay, do it again. Find the midpoints of these four segments, labeling them O_{21}, O_{22}, O_{23}, and O_{24}. These new midpoints give us semicircles of radius $\frac{r_2}{2} = \frac{r}{4}$ and

$$\alpha_{21} = \alpha_{22} = \alpha_{23} = \alpha_{24} = \pi r_3 = \frac{\pi r}{4} \,.$$

Just as before the length of the curve following along the four semicircles will be

$$d_{c3} = \sum_{k=1}^{4} a_{2k} = 4 \frac{\pi r}{4} = \pi r \text{, again!}$$

This then gives us the idea to continue this process, each time the radius will be half of the previous radius and at the same time the number of semicircles doubles. Thus, at the nth stage we have



$$r_n = \frac{r}{2^{n-1}}$$

$$\alpha_{nk} = \pi r_n,\ k = 1,\dots,2^{n-1}$$

$$d_{Cn} = \sum_{k=1}^{2^{n-1}} \alpha_{nk} = 2^{n-1}\frac{\pi r}{2^{n-1}} = \pi r.$$

Therefore the trip along the semicircles is always of length πr.

Note that each stage of this process takes the path closer and closer to the diameter. In fact, it should not be too hard to believe and not too hard for the reader to show that given $\epsilon > 0$ there is an $N > 0$ so that the maximum horizontal distance from the diameter to the curve made of the $\alpha_N k$ is less than ϵ. (Take N so that $r/2^N < \epsilon$ should work.) This means that the curve so constructed is converging to the diameter.

If length is preserved then the length of the diameter and the length of the limiting curve must be the same. These lengths have been constant all along. Therefore we would have to have that:

$$\pi r = 2r \text{ or } \pi = 2.$$

This seems to be a problem. Thus, length cannot be preserved under pointwise convergence. It can get worse than this, by all means.

Dr. David Royster
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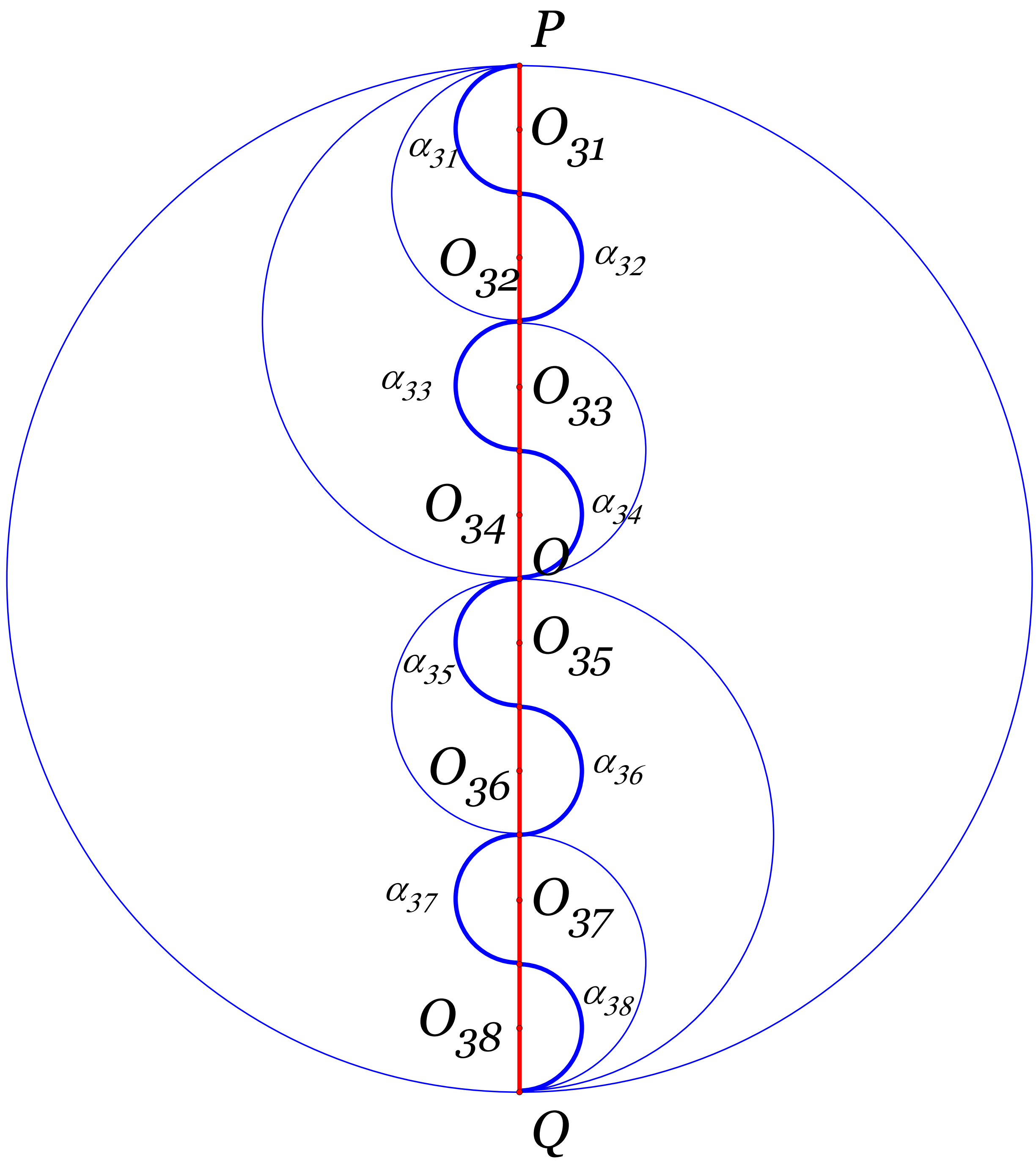
P
O_{31}
α_{31}
O_{32}
α_{32}
α_{33}
O_{33}
O_{34}
α_{34}
O
O_{35}
α_{35}
O_{36}
α_{36}
α_{37}
O_{37}
O_{38}
α_{38}
Q