Appendix A

An Overview of Babylonian Mathematics

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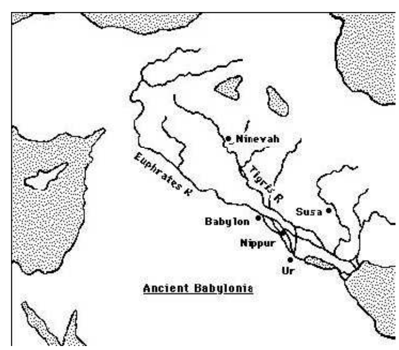
A.1 Introduction to Babylonian Mathematics

The Babylonians lived in Mesopotamia, a fertile crescent between the Tigris and Euphrates rivers. Here is a map of the region where the civilization flourished.



The region had been the center of the Sumerian civilization which flourished before 3500 BC. This was an advanced civilization building cities and supporting the people with irrigation systems, a legal system, administration, and even a postal service. Writing developed and counting was based on a sexagesimal system, that is base 60. Around 2300 BC the Akkadians invaded the area and for some time the less civilized culture of the Akkadians mixed with the more advanced culture of the Sumerians. The Akkadians invented the abacus as a tool for counting and they developed somewhat clumsy methods of arithmetic with addition, subtraction, multiplication and division all playing a part. The Sumerians, however, revolted against Akkadian rule and by 2100 BC they were back in control.

The Babylonian civilization, whose mathematics is the subject of this article, replaced that of the Sumerians starting around 2000 BC. The Babylonians were a Semitic people who invaded Mesopotamia, defeated the Sumerians and by about 1900 BC established their capital at Babylon.

The Sumerians had developed an abstract form of writing based on cuneiform (i.e. wedge-shaped) symbols. Their symbols were written on wet clay tablets which were baked in the hot sun. Many thousands of these tablets have survived to this day. It was the use of a stylus on a clay medium that led to the use of cuneiform symbols since curved lines could not easily be drawn. The later Babylonians adopted the same style of cuneiform writing on clay tablets. A picture of one of their tablets Babylonian mathematics is to the right.

Many of the tablets concern topics which are fascinating, although they do not contain deep mathematics. For example we mentioned above the irrigation systems of the early civilizations in Mesopotamia. Muroi[1] writes:

It was an important task for the rulers of Mesopotamia to dig canals and to maintain them, because canals were not only necessary for irrigation but also useful for the transport of goods and armies. The rulers or high government officials must have ordered Babylonian mathematicians to calculate the number of workers and days necessary for the building of a canal, and to calculate the total expenses of wages of the workers.

There are several Old Babylonian mathematical texts in which various quantities concerning the digging of a canal are asked for. They are YBC 4666, 7164, and VAT 7528, all of which are written in Sumerian ..., and YBC 9874 and BM 85196, No. 15, which are written in Akkadian ... . From the mathematical point of view these problems are comparatively simple ...

The Babylonians had an advanced number system, in some ways more advanced than our present systems. It was a positional system with a base of 60 rather than the system with base 10.

The Babylonians divided the day into 24 hours, each hour into 60 minutes, each minute into 60 seconds. This form of counting has survived for 4000 years. To write 5h 25' 30", i.e. 5 hours, 25 minutes, 30 seconds, is just to write the sexagesimal fraction, 5 25/60 30/3600. We adopt the notation 5;25,30 for this sexagesimal number. As a base 10 fraction the sexagesimal number 5;25,30 is 5 4/10 2/100 5/1000 which is written as 5.425 in decimal notation.

Perhaps the most amazing aspect of the Babylonian's calculating skills was their construction of tables to aid calculation. Two tablets found at Senkerah on the Euphrates in 1854 date from 2000 BC. They give squares of the numbers up to 59 and cubes of the numbers up to 32. The table gives $8^2 = 1,4$ which stands for

$$8^2 = 1,4 = 1 \times 60 + 4 = 64$$

[1]K Muroi, Small canal problems of Babylonian mathematics, *Historia Sci.* (2) 1 (3) (1992), 173-180

and so on up to $59^2 = 58, 1(= 58 \times 60 + 1 = 3481)$. The Babylonians used the formula

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$

to make multiplication easier. Even better is their formula

$$ab = \frac{(a+b)^2 - (a-b)^2}{4}$$

which shows that a table of squares is all that is necessary to multiply numbers, simply taking the difference of the two squares that were looked up in the table then taking a quarter of the answer.

Division is a harder process. The Babylonians did not have an algorithm for long division. Instead they based their method on the fact that

$$\frac{a}{b} = a \times \frac{1}{b}$$

so all that was necessary was a table of reciprocals. We still have their reciprocal tables going up to the reciprocals of numbers up to several billion. Of course these tables are written in their numerals, but using the sexagesimal notation we introduced above, the beginning of one of their tables would look like:

2	$0;30$	$1/2 = 0 + 30/60$
3	$0;20$	$1/3 = 0 + 20/60$
4	$0;15$	$1/4 = 0 + 15/60$
5	$0;12$	$1/5 = 0 + 12/60$
6	$0;10$	$1/6 = 0 + 10/60$
8	$0;7,30$	$1/7 = 0 + 7/60 + 30/3600$
9	$0;6,40$	$1/9 = 0 + 6/60 + 40/3600$
10	$0;6$	$1/10 = 0 + 6/60$
12	$0;5$	$1/12 = 0 + 5/60$
15	$0;4$	$1/15 = 0 + 4/60$
16	$0;3,45$	$1/16 = 0 + 3/60 + 45/3600$
18	$0;3,20$	$1/18 = 0 + 3/60 + 20/3600$
20	$0;3$	$1/20 = 0 + 3/60$
24	$0;2,30$	$1/24 = 0 + 2/60 + 30/3600$
25	$0;2,24$	$1/25 = 0 + 2/60 + 24/3600$
27	$0;2,13,20$	$1/27 = 0 + 2/60 + 13/3600 + 20/216000$

Now the table had gaps in it since 1/7, 1/11, 1/13, *etc.* are not finite base 60 fractions. This did not mean that the Babylonians could not compute 1/13. They would write

$$\frac{1}{13} = \frac{7}{91} = 7 \times \frac{1}{91} \sim 7 \times \frac{1}{90}$$

and these values, for example 1/90, were given in their tables. In fact there are fascinating glimpses of the Babylonians coming to terms with the fact that division by 7 would lead to an infinite sexagesimal fraction. A scribe would give a number close to 1/7 and then write statements such as

... an approximation is given since 7 does not divide.[2]

Babylonian mathematics went far beyond arithmetical calculations. We now examine some algebra which the Babylonians developed, particularly problems Babylonian mathematics which led to equations and their solution. The Babylonians were famed as constructors of tables. Now these could be used to solve equations. For example they constructed tables for n^3+n^2 then, with the aid of these tables, certain cubic equations could be solved. For example, consider the equation

$$ax^3 + bx^2 = c.$$

Note that we are using modern notation, and nothing like this symbolic representation existed in Babylonian times. The Babylonians could handle numerical examples of such equations. They did this by using certain rules, which indicates that they did have the concept of a typical problem of a given type and a typical method to solve it. For example in the above case they would (in modern notation) multiply the equation by a^2 and divide it by b^3 to get

$$\left(\frac{ax}{b}\right)^3 + \left(\frac{ax}{b}\right)^2 = \frac{ca^2}{b^3}.$$

Setting $y = ax/b$ gives the equation $y^3 + y^2 = ca^2/b^3$ which could now be solved by looking up the $n^3 + n^2$ table for the value of n satisfying $n^3 + n^2 = ca^2/b^3$. When a solution was found for y then x was found by $x = by/a$. We cannot stress too much that all this was done without algebraic notation.

Again a table would have been looked up to solve the linear equation $ax = b$. They would consult the $1/n$ table to find $1/a$ and then multiply the sexagesimal number given in the table by b. An example of a problem of this type is the following.

> *Suppose, writes a scribe,* 2/3 *of* 2/3 *of a certain quantity of barley is taken, 100 units of barley are added and the original quantity recovered.*

The problem posed by the scribe is to find the quantity of barley. The solution given by the scribe is to compute $0;40$ times $0;40$ to get $0;26,40$. Subtract this from $1;00$ to get $0;33,20$. Look up the reciprocal of $0;33,20$ in a table to get $1;48$. Multiply $1;48$ by $1,40$ to get the answer $3,0$.

It is not that easy to understand these calculations by the scribe unless we translate them into modern algebraic notation. We have to solve

$$\frac{2}{3} \times \frac{2}{3}x + 100 = x$$

[2]G. G. Joseph, *The crest of the peacock* (London, 1991)

which is, as the scribe knew, equivalent to solving

$$\left(1 - \frac{4}{9}\right) x = 100.$$

This is why the scribe computed $\frac{2}{3} \times \frac{2}{3}$ subtracted the answer from 1 to get $1 - \frac{4}{9}$, then looked up $1/(1 - \frac{4}{9})$ and so x was found from $1/(1 - \frac{4}{9})$ multiplied by 100 giving 180 (which is $1;48$ times $1,40$ to get $3,0$ in sexagesimal).

To solve a quadratic equation the Babylonians essentially used the standard formula. They considered two types of quadratic equation, namely $x^2 + bx = c$ and $x^2 - bx = c$ where here b, c were positive but not necessarily integers. The form that their solutions took was, respectively

$$x = \left(\left(\frac{b}{2}\right)^2 + c\right) - \frac{b}{2} \text{ and } \left(\left(\frac{b}{2}\right)^2 + c\right) + \frac{b}{2}.$$

Notice that in each case this is the positive root from the two roots of the quadratic and the one which will make sense in solving "real" problems. For example problems which led the Babylonians to equations of this type often concerned the area of a rectangle. For example if the area is given and the amount by which the length exceeds the width is given, then the width satisfies a quadratic equation and then they would apply the first version of the formula above.

A problem on a tablet from Babylonian times states that the area of a rectangle is $1,0$ and its length exceeds its width by 7. The equation $x^2 + 7x = 1,0$ is, of course, not given by the scribe who finds the answer as follows.

> Compute half of 7, namely $3;30$, square it to get $12;15$. To this the scribe adds $1,0$ to get $1;12,15$. Take its square root (from a table of squares) to get $8;30$. From this subtract $3;30$ to give the answer 5 for the breadth of the triangle.

Notice that the scribe has effectively solved an equation of the type $x^2 + bx = c$ by using $x = ((b/2)^2 + c) - (b/2)$. Berriman[3] gives 13 typical examples of problems leading to quadratic equations taken from Old Babylonian tablets.

If problems involving the area of rectangles lead to quadratic equations, then problems involving the volume of rectangular excavation (a "cellar") lead to cubic equations. The clay tablet BM 85200+ containing 36 problems of this type, is the earliest known attempt to set up and solve cubic equations[4]. Of course the Babylonians did not reach a general formula for solving cubics. This would not be found for well over three thousand years.

[3] A. E. Berriman, The Babylonian quadratic equation, *Math. Gaz.* 40 (1956), 185-192

[4] J. Hoyrup, The Babylonian cellar text BM 85200+ VAT 6599. Retranslation and analysis, *Amphora* (Basel, 1992), 315-358

A.2 Pythagoras's theorem in Babylonian mathematics

In this section we examine four Babylonian tablets which all have some connection with Pythagoras's theorem. Certainly the Babylonians were familiar with Pythagoras' theorem. A translation of a Babylonian tablet which is preserved in the British museum goes as follows

> *4 is the length and 5 the diagonal. What is the breadth?*
> *Its size is not known.*
> *4 times 4 is 16.*
> *5 times 5 is 25.*
> *You take 16 from 25 and there remains 9.*
> *What times what shall I take in order to get 9?*
> *3 times 3 is 9.*
> *3 is the breadth.*

All the tablets come from roughly the same period, namely that of the Old Babylonian Empire which flourished in Mesopotamia between 1900 BC and 1600 BC.

The four tablets which interest us here we will call the Yale tablet YBC 7289, Plimpton 322, the Susa tablet, and the Tell Dhibayi tablet. Let us say a little about these tablets before describing the mathematics which they contain.

The Yale tablet YBC 7289 which we describe is one of a large collection of tablets held in the Yale Babylonian collection of Yale University. It consists of a tablet on which a diagram appears. The diagram is a square of side 30 with the diagonals drawn in. The tablet and its significance was first discussed in Neugebauer[5] and in Fowler and Robson[6].

Plimpton 322 is the tablet numbered 322 in the collection of G A Plimpton housed in Columbia University. The top left hand corner of the tablet is damaged as and there is a large chip out of the tablet around the middle of the right hand side. Its date is not known accurately but it is put at between 1800 BC and 1650 BC. It is thought to be only part of a larger tablet, the remainder of which has been destroyed, and at first it was thought to be a record of commercial transactions. However Neugebauer and Sachs gave a new interpretation and since then it has been the subject of a huge amount of interest.

The Susa tablet was discovered at the present town of Shush in the Khuzistan region of Iran. The town is about 350 km from the ancient city of Babylon. W. K. Loftus identified this as an important archaeological site as early as 1850 but excavations were not carried out until much later. This particular tablet investigates

[5] O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts* (New Haven, CT, 1945).

[6] D Fowler and E Robson, Square root approximations in old Babylonian mathematics: YBC 7289 in context, Historia Math. 25 (4) (1998), 366-378

how to calculate the radius of a circle through the vertices of an isosceles triangle.

Finally the Tell Dhibayi tablet was one of about 500 tablets found near Baghdad by archaeologists in 1962. Most relate to the administration of an ancient city which flourished in the time of Ibalpiel II of Eshunna and date from around 1750 BC. The particular tablet in which we are interested is not one relating to administration but one which presents a geometrical problem which asks for the dimensions of a rectangle whose area and diagonal are known.

Before looking at the mathematics contained in these four tablets we should say a little about their significance in understanding the scope of Babylonian mathematics. First, you must be careful not to read into early mathematics any ideas which we can see clearly today yet which were never in the mind of that author. Conversely, we must be careful not to underestimate the significance of the mathematics just because it has been produced by mathematicians who thought very differently from today's mathematicians. As a final comment on what these four tablets tell us of Babylonian mathematics we must be careful to realize that almost all of the mathematical achievements of the Babylonians, even if they were all recorded on clay tablets, have been lost and, even if these four may be seen as especially important among those surviving, they may not represent the best of Babylonian mathematics.

There is no problem understanding what the Yale tablet YBC 7289 is about — a diagram is on the right.

It has on it a diagram of a square with 30 on one side, the diagonals are drawn in and near the center is written $1, 24, 51, 10$ and $42, 25, 35$. These are numbers are written in Babylonian numerals to base 60. The Babylonian numbers are always ambiguous and no indication occurs as to where the integer part ends and the fractional part begins. Assuming that the first number is $1; 24, 51, 10$ then converting this to a decimal gives 1.414212963 while $\sqrt{2} = 1.414213562$. Calculating $30 \times 1; 24, 51, 10$ gives $42; 25, 35$ which is the second number. The diagonal of a square of side 30 is found by multiplying 30 by the approximation to $\sqrt{2}$.

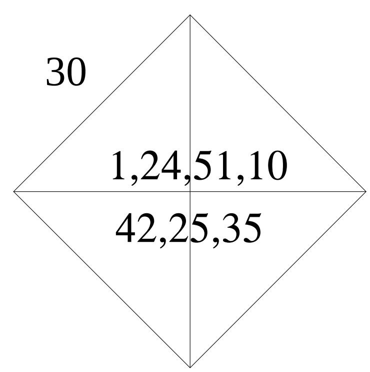


Figure A.1: Yale Tablet diagram

This shows a nice understanding of Pythagoras's theorem. However, even more significant is the question how the Babylonians found this remarkably good approximation to $\sqrt{2}$. Several authors[7] conjecture that the Babylonians used a method equivalent to Heron's method. The suggestion is that they started with a guess, say x. They then found $e = x^2 - 2$ which is the error. Then

[7] R. Calinger, *A conceptual history of mathematics* (Upper Straddle River, N. J., 1999) and G. G. Joseph, *The crest of the peacock* (London, 1991)

$$\begin{aligned}\left(x - \frac{e}{2x}\right)^2 &= x^2 - e + \left(\frac{e}{2x}\right)^2 \\ &= 2 + \left(\frac{e}{2x}\right)^2\end{aligned}$$

and they had a better approximation since if e is small then $(e/2x)^2$ will be very small. Continuing the process with this better approximation to $\sqrt{2}$ yields a still better approximation and so on. In fact as Joseph points out, you need only two steps of the algorithm if you start with $x = 1$ to obtain the approximation $1;24,51,10$.

This is certainly possible and the Babylonians' understanding of quadratics adds some weight to the claim. However there is no evidence of the algorithm being used in any other cases and its use here must remain no more than a fairly remote possibility.

E. F. Robertson suggests an alternative. The Babylonians produced tables of squares, in fact their whole understanding of multiplication was built round squares, so perhaps a more obvious approach for them would have been to make two guesses, one high and one low say a and b. Take their average $(a + b)/2$ and square it. If the square is greater than 2 then replace b by this better bound, while if the square is less than 2 then replace a by $(a + b)/2$. Continue with the algorithm. Now this certainly takes many more steps to reach the sexagesimal approximation $1;24,51,10$. In fact starting with $a = 1$ and $b = 2$ it takes 19 steps as the table below shows:

STEP	DECIMAL APPROX	SEXAGESIMAL APPROX
1	1.500000000	1;29,59,59
2	1.250000000	1;14,59,59
3	1.375000000	1;22,29,59
4	1.437500000	1;26,14,59
5	1.406250000	1;24,22,29
6	1.421875000	1;25,18,44
7	1.414062500	1;24,50,37
8	1.417968750	1;25,04,41
9	1.416015625	1;24,57,39
10	1.415039063	1;24,54,08
11	1.414550781	1;24,52,22
12	1.414306641	1;24,51;30
13	1.414184570	1;24,51;03
14	1.414245605	1;24,51;17
15	1.414215088	1;24,51;10
16	1.414199829	1;24,51;07
17	1.414207458	1;24,51;08
18	1.414211273	1;24,51;09
19	1.414213181	1;24,51;10

The Babylonians were not afraid to undertake a computation and they may have been prepared to continue this straightforward calculation until the answer was correct to the third sexagesimal place.

Next we look again at Plimpton 322 The tablet has four columns with 15 rows. The last column is the simplest to understand for it gives the row number and so contains 1, 2, 3, ... , 15. The remarkable fact which Neugebauer and Sachs pointed out is that in every row the square of the number c in column 3 minus the square of the number b in column 2 is a perfect square, say h.

$$c^2 - b^2 = h^2$$

So the table is a list of Pythagorean integer triples. Now this is not quite true since Neugebauer and Sachs believe that the scribe made four transcription errors, two in each column and this interpretation is required to make the rule work. The errors are readily seen to be genuine errors, however, for example $8, 1$ has been copied by the scribe as $9, 1$. The first column is harder to understand, particularly since damage to the tablet means that part of it is missing. However, using the above notation, it is seen that the first column is just $(c/h)^2$.

So far so good, but if one were writing down Pythagorean triples one would find much easier ones than those which appear in the table. For example the Pythagorean triple $(3, 4, 5)$ does not appear, nor does $(5, 12, 13)$ and in fact the smallest Pythagorean triple which does appear is $(45, 60, 75)$ (15 times $(3, 4, 5)$). Also the rows do not appear in any logical order except that the numbers in column 1 decrease regularly. The puzzle then is how the numbers were found and why are these particular Pythagorean triples are given in the table. Several historians (see for example Calinger) have suggested that column 1 is connected with the secant function. However, as Joseph comments

This interpretation is a trifle fanciful.

Zeeman has made a fascinating observation. He has pointed out that if the Babylonians used the formulas $h = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$ to generate Pythagorean triples then there are exactly 16 triples satisfying $n \leq 60$, $30° \leq t \leq 45°$, and $tan^2 t = h^2/b^2$ having a finite sexagesimal expansion (which is equivalent to m, n, b having 2, 3, and 5 as their only prime divisors). Now 15 of the 16 Pythagorean triples satisfying Zeeman's conditions appear in Plimpton 322. Is it the earliest known mathematical classification theorem? Are we now reading too much into the mathematics of the Babylonians, though.

To give a fair discussion of Plimpton 322 we should add that not all historians agree that this tablet concerns Pythagorean triples. For example Exarchakos[8] claims that the tablet is connected with the solution of quadratic equations and has nothing

[8] T. G. Exarchakos, Babylonian mathematics and Pythagorean triads, *Bull. Greek Math. Soc.* 37 (1995), 29-47

to do with Pythagorean triples; "*we prove that in this tablet there is no evidence whatsoever that the Babylonians knew the Pythagorean theorem and the Pythagorean triads.*" According to E.F. Robertson there are numerous tablets which show that the Babylonians of this period had a good understanding of Pythagoras's theorem. Other authors, although accepting that Plimpton 322 is a collection of Pythagorean triples, have argued that they had, as Viola[9] writes a practical use in giving a; "*general method for the approximate computation of areas of triangles.*"

The Susa tablet sets out a problem about an isosceles triangle with sides 50, 50 and 60. The problem is to find the radius of the circle through the three vertices.

Here we have labeled the triangle A, B, C and the center of the circle is O. The perpendicular AD is drawn from A to meet the side BC. Now the triangle $\triangle ABD$ is a right angled triangle so, using Pythagoras' theorem $AD^2 = AB^2 - BD^2$, so $AD = 40$. Let the radius of the circle be x. Then $AO = OB = x$ and $OD = 40 - x$. Using Pythagoras's theorem again on the triangle $\triangle OBD$ we have $x^2 = OD^2 + DB^2$. So $x^2 = (40 - x)^2 + 30^2$ giving $x^2 = 40^2 - 80x + x^2 + 30^2$ and so $80x = 2500$ or, in sexagesimal, $x = 31; 15$.

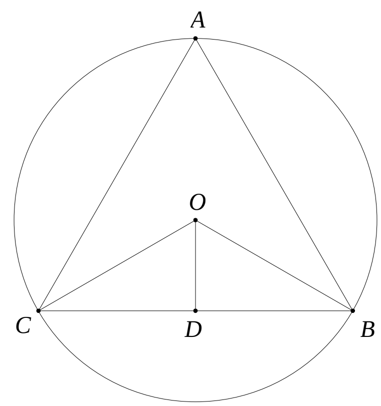


Figure A.2: Susa tablet diagram

Finally consider the problem from the Tell Dhibayi tablet. It asks for the sides of a rectangle whose area is $0; 45$ and whose diagonal is $1; 15$. Now this to us is quite an easy exercise in solving equations. If the sides are x and y we have $xy = 0.75$ and $x^2 + y^2 = (1.25)^2$. We would substitute $y = 0.75/x$ into the second equation to obtain a quadratic in x^2 which is easily solved. This however is not the method of solution given by the Babylonians and really that is not surprising since it rests heavily on our algebraic understanding of equations. The way the Tell Dhibayi tablet solves the problem is actually much more interesting than the modern method.

Here is the method from the Tell Dhibayi tablet. We preserve the modern notation x and y as each step for clarity but we do the calculations in sexagesimal notation (as of course does the tablet). Compute $2xy = 1; 30$. Subtract from $x^2 + y^2 = 1; 33, 45$ to get $x^2 + y^2 - 2xy = 0; 3, 45$. Take the square root to obtain $x - y = 0; 15$. Divide by 2 to get $(x - y)/2 = 0; 7, 30$. Divide $x^2 + y^2 - 2xy = 0; 3, 45$ by 4 to get $x^2/4 + y^2/4 - xy/2 = 0; 0, 56, 15$. Add $xy = 0; 45$ to get $x^2/4 + y^2/4 + xy/2 = 0; 45, 56, 15$. Take the square root to obtain $(x + y)/2 = 0; 52, 30$. Add $(x + y)/2 = 0; 52, 30$ to $(x - y)/2 = 0; 7, 30$ to get $x = 1$. Subtract $(x - y)/2 = 0; 7, 30$ from $(x + y)/2 = 0; 52, 30$ to get $y = 0; 45$. Hence the rectangle has sides $x = 1$ and $y = 0; 45$. Remember that this piece of mathematics is 3750 years old.

[9]T Viola, On the list of Pythagorean triples ("Plimpton 322") and on a possible use of it in old Babylonian mathematics (Italian), *Boll. Storia Sci. Mat.* 1 (2) (1981), 103-132