Appendix H

The Lagrange Inversion Theorem

In mathematical analysis, the Lagrange Inversion theorem gives the Taylor series expansion of the inverse function of an analytic function. Suppose that w and z is implicitly related by an equation of the form

$$f(w) = z$$

where f is analytic at a point a and $f'(a) \neq 0$. Then it is possible to invert or solve the equation for w

$$w = g(z)$$

where g is analytic at the point $b = f(a)$.

When this is done to derive the series expansion for g, it is also called **reversion of series**. The series expansion of g is given by

$$g(z) = a + \sum_{n=1}^{\infty} \frac{d^{n-1}}{(dw)^{n-1}} \left(\frac{w-a}{f(w)-b} \right)^n \Bigg|_{w=a} \frac{(z-b)^n}{n!}.$$

This formula can be used to find the Taylor series of the Lambert W function (by setting $f(w) = we^w$ and $a = b = 0$).

In this case we have to compute the $(n-1)$st derivative of $(w/f(w))^n$ at $w = 0$. Since $f(w) = we^w$, we must compute the $(n-1)$st derivative of e^{-nw} at $w = 0$. The main thing here is to keep track of the signs.

$$\frac{d^{n-1}e^{-nw}}{dw^{n-1}} = (-n)^{n-1}e^{-nw} \Rightarrow \frac{d^n e^{-nw}}{dw^n}\Bigg|_{w=0} = (-n)^{n-1}.$$

Therefore, the series expansion for the Lambert W function is as reported:

$$W(x) = \sum_{n \geq 1} \frac{(-n)^{n-1}}{n!} x^n.$$

The formula is also valid for formal power series and can be generalized in various ways. If it can be formulated for functions of several variables, it can be extended to provide a ready formula for $F(g(z))$ for any analytic function F, and it can be generalized to the case $f'(a) = 0$, where the inverse g is a multi-valued function.

The theorem was proved by Lagrange and generalized by Bürmann, both in the late 18th century. There is a straightforward derivation using complex analysis and contour integration (the complex formal power series version is clearly a consequence of knowing the formula for polynomials, so the theory of analytic functions may be applied).

Series reversion is the computation of the coefficients of the inverse function given those of the forward function. For a function expressed in a series with no constant term (i.e., $a_0 = 0$) as

$$y = a_1 x + a_2 x^2 + a_3 x^3 + \dots,$$

the series expansion of the inverse series is given by

$$x = A_1 y + A_2 y^2 + A_3 y^3 + \dots.$$

By plugging this second series into the first, the following equation is obtained

$$y = a_1 A_1 y + (a_2 A_1^2 + a_1 A_2) y^2 + (a_3 A_1^3 + 2a_2 A_1 A_2 + a_1 A_3) y^3 + (3a_3 A_1^2 A_2 + a_2 A_2^2 + a_2 A_1 A_3) y^4 + \dots,$$

Equating coefficients then gives

$$\begin{aligned}
A_1 &= a_1^{-1} \\
A_2 &= -a_1^{-3} a_2 \\
A_3 &= a_1^{-5}(2a_2^2 - a_1 a_3) \\
A_4 &= a_1^{-7}(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3) \\
A_5 &= a_1^{-9}(6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21 a_1 a_2^2 a_3) \\
A_6 &= a_1^{-11}(7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84 a_1 a_2^3 a_3 - a_1^4 a_6 - 28 a_1^2 a_2 a_3^2 - 42 a_2^5 - 28 a_1^2 a_2^2 a_4) \\
A_7 &= a_1^{-13}(8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 4a_1^4 a_4^2 + 120 a_1^2 a_2^3 a_4 + 180 a_1^2 a_2^2 a_3^2 + 132 a_2^6 - a_1^4 a_7 - \\
&\qquad 36 a_1^3 a_2^2 a_5 - 72 a_1^3 a_2 a_3 a_4 - 12 a_1^3 a_3^3 - 330 a_1 a_2^4 a_3)
\end{aligned}$$

As an example we know that

$$y = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

Thus, $a_n = 1/n!$. We will use Series Reversion to find the series for $x = \ln y$.

$$\begin{aligned}
A_1 &= a_1^{-1} = 1\\
A_2 &= -a_1^{-3}a_2\\
&= -\frac{1}{2}\\
A_3 &= a_1^{-5}(2a_2^2 - a_1a_3)\\
&= 2\times\frac{1}{4} - \frac{1}{6} = \frac{1}{3}\\
A_4 &= a_1^{-7}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3)\\
&= 5\times\frac{1}{2}\times\frac{1}{6} - \frac{1}{24} - 5\times\frac{1}{8} = -\frac{1}{4}\\
A_5 &= a_1^{-9}(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3)\\
&= 6\times\frac{1}{2}\times\frac{1}{24} + 3\times\frac{1}{36} + 14\times\frac{1}{16} - \frac{1}{120} - 21\times\frac{1}{4}\times\frac{1}{6} = \frac{1}{5}
\end{aligned}$$

and so forth. Note that this does give us the power series for $\ln(1+x)$, which you will remember is

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}.$$