Chapter 1

A Brief Background to Numbers and How we Got Here

1.1 Introduction

How do we think about numbers and mathematics? What is a number? Even more so, what is mathematics? We seem to have 5000 year history of number and mathematics, yet are we any different from the Greek or Egyptian or Babylonian or Vedic mathematicians millennia ago?

Counting goes back very far into prehistory. Paleontologists have discovered drawings indicating some knowledge of mathematics on ochre rocks in a cave in South Africa that had scratched geometric patterns dating back to about 70,000 B.C. There are other artifacts from Africa and France, dated between 35,000 BC and 20,000 BC that indicate an attempt to "quantify" time.

The Ishango bone is a bone tool, dated to the Upper Paleolithic era, about 20,000 to 18,000 BC. It is a dark brown length of bone, with a sharp piece of quartz affixed to one end. It was first thought to be a tally stick, as it has a series of tally marks carved in three columns running the length of the tool, but some scientists have suggested that the groupings of notches indicate that there may be a mathematical understanding that goes beyond counting. One interpretation is that the bone is the earliest known demonstration[1] of sequences of prime numbers and Ancient Egyptian multiplication. It was found in 1950 by Belgian Jean de Heinzelin de Braucourt while exploring what was then the Belgian Congo. It was discovered in the African area of Ishango, which was centered near the headwaters of the Nile River at Lake Edward (now on the border between modern-day Uganda and Congo). The lakeside Ishango population of 20,000 years ago may have been one of the first counting societies, but it lasted only a few hundred years before being buried by a volcanic eruption. The

[1]Williams, Scott W. (2005). *An Old Mathematical Object*. **Mathematicians of the African Diaspora**. SUNY Buffalo mathematics department. Retrieved on 2006-08-15. [`http://www.math.buffalo.edu/mad/Ancient-Africa/ishango.html`]

artifact was first estimated to originate between 9000 BC and 6500 BC. However, the dating of the site where it was discovered was re-evaluated, and is now believed to be more than 20,000 years old.

Several tally sticks predate the Ishango bone, and cuts on sticks or bones have been found worldwide. The Lebombo bone, a 37000-year-old baboon fibula was found in Swaziland. A 32000-year-old wolf tibia with 57 notches, grouped in fives, was found in Czechoslovakia in 1937.

Some claim that Megalithic monuments from as early as the 5th millennium BC in Egypt, and then subsequently England and Scotland from the 3rd millennium BC[2], incorporate geometric ideas such as circles, ellipses, and Pythagorean triples in their design, as well as a possible understanding of the measurement of time based on the movement of the stars. From about 3100 BC Egyptians introduced the earliest known decimal system, allowing indefinite counting by way of introducing new symbols.

The earliest known mathematics in ancient India dates back to circa 3000-2600 BC in the Indus Valley Civilization (Harappan civilization) of North India and Pakistan, which developed a system of uniform weights and measures that used decimal fractions, an advanced brick technology which used ratios, streets laid out in perfect right angles, and a number of geometrical shapes and designs, including cuboids, barrels, cones, cylinders, and drawings of concentric and intersecting circles and triangles. Mathematical instruments which were discovered include an accurate decimal ruler with small and precise subdivisions, a shell instrument that served as a compass to measure angles on plane surfaces or in horizon in multiples of 40360 degrees, a shell instrument used to measure 812 whole sections of the horizon and sky, and an instrument for measuring the positions of stars for navigational purposes. The Indus script has not yet been deciphered; hence very little is known about the written forms of Harappan mathematics. Archeological evidence has led some historians to believe that this civilization used a base 8 numeral system and possessed knowledge of the ratio of the length of the circumference of the circle to its diameter, thus a value of π.[3]

We can tell that the Egyptians and the Babylonians (as well as the Indians and Chinese) built their knowledge of mathematics from their observations of nature. Earlier historians assumed that a great majority of the knowledge was *ad hoc* and was tested by trial-and-error.

Clay tablets from the Sumerian (2100 BC) and the Babylonian cultures (1600 BC) include tables for computing products, reciprocals, squares, square roots, and other mathematical functions useful in financial calculations. Babylonians were able

[2] Thom, Alexander and Archie Thom, "The metrology and geometry of Megalithic Man," pp 132-151 in C.L.N. Ruggles, ed., *Records in Stone: Papers in memory of Alexander Thom*, (Cambridge: Cambridge Univ. Pr., 1988)

[3] Pearce, Ian G. (2002). *Early Indian culture - Indus civilization.* Indian Mathematics: Redressing the balance. School of Mathematical and Computational Sciences University of St Andrews. Retrieved on 2006-08-15. [`http://www-groups.dcs.st-and.ac.uk/ history/Miscellaneous/Pearce/Lectures/Ch3.html`]

to compute areas of rectangles, right and isosceles triangles, trapezoids and circles. They computed the area of a circle as the square of the circumference divided by twelve. The Babylonians were also responsible for dividing the circumference of a circle into 360 equal parts. They also used the Pythagorean Theorem (long before Pythagoras), performed calculations involving ratio and proportion, and studies the relationships between the elements of various triangles. There is also evidence that the Babylonian mathematicians were able to solve more complicated quadratic equations, and did this sometimes for the ”sheer enjoyment” of solving the equations. This is a first step for moving mathematics away from the utilitarian tool that it seems to have been.

Accounts that we now have about the early history of mathematics in India and China indicate that they too had a quite sophisticated knowledge of mathematics and it was also pushed beyond being merely utilitarian.

From what we seem to know about the history of mathematics it seems that Thales of Miletus is the first to take mathematics from the inductive (observed and recorded) to the *deductive*. Thales precedes the Pythagoreans and is credited with a number of results in geometry. The Greek philosophy on mathematics was strongly influenced by their study of geometry. At one time, the Greeks held the opinion that 1 (one) was not a number, but rather a unit of arbitrary length so that 3, for example, represented 3 such units and truly was a number. These views come from the heavily geometric straight-edge-and-compass viewpoint of the Greeks: just as lines drawn in a geometric problem are measured in proportion to the first arbitrarily drawn line, so too are the numbers on a number line measured in proportional to the arbitrary first ”number” or ”one.”

These earlier Greek ideas of number were later upended by the discovery of the irrationality of the square root of two. According to legend, Hippasus, a disciple of Pythagoras showed that the diagonal of a unit square was incommensurable with its (unit-length) edge: in other words he proved there was no existing (rational) number that accurately depicts the proportion of the long diagonal of the unit square to its edge. Fellow Pythagoreans were so traumatized by this discovery that they would murder Hippasus to stop him from spreading his heretical idea.

Thus it was the Pythagoreans, though, that changed the nature of “number.” They and Plato noted that the conclusions that they reached “deductively” agreed to a remarkable extent to the results from observation and inductive inference. Philosophically they were unable to account for this agreement so they were led to regard mathematics as the study of an ultimate, eternal reality rather than a branch of logic or a tool of science and technology. Their view is reflected in the Pythagorean dictum “All is number.”

This means that mathematical objects are ideal objects without the restrictions of physical properties. Thus, a square is an idealized square sitting out in an ideal world. All mathematicians have to do is to “discover” the properties of these idealized objects.

This falls into the realm of *mathematical realism*. Mathematical realism, like realism in general, holds that mathematical entities exist independently of the human mind. Thus humans do not invent mathematics, but rather discover it, and any other intelligent beings in the universe would presumably do the same. In this point of view, there is really one sort of mathematics that can be discovered: Triangles, for example, are real entities, not the creations of the human mind. Many working mathematicians have been mathematical realists; they see themselves as discoverers of naturally occurring objects. Within mathematical realism, there are distinctions depending on what sort of existence one takes mathematical entities to have, and how we know about them.

What sort of a philosophy of mathematics does our state Standard Course of Study assume? How about Piagetian constructivism? Social constructivism?

Platonism is a form of realism that suggests that mathematical entities are abstract, have no spatio-temporal or causal properties, and are eternal and unchanging. This is often claimed to be the naïve view most people have of numbers. The term Platonism is used because such a view is seen to parallel Plato's belief in a *World of Ideas*, an unchanging ultimate reality that the everyday world can only imperfectly approximate. Plato's view probably derives from the Pythagoreans who believed that the world was, quite literally, built up by the numbers. The major problem of mathematical platonism is this: precisely where and how do the mathematical entities exist, and how do we know about them? Is there a world, completely separate from our physical one, which is occupied by the mathematical entities? How can we gain access to this separate world and discover truths about the entities? Gödel's platonism postulates a special kind of mathematical intuition that lets us perceive mathematical objects directly. Davis and Hersh have suggested in their book *The Mathematical Experience* that most mathematicians act as though they are Platonists, even though, if pressed to defend the position carefully, they may retreat to formalism.

In the Middle Ages we see a "Scholastic" view prevailing. This is the view that the universe is "tidy" and simply intelligible. As we enter the 14th century the realization that the former qualitative view of motion and variation could be better replaced by a quantitative study. As mathematics began to move into the Renaissance in Italy and there was a revival in the Platonic views, we see in the 15th and 16th centuries a renewal of the conviction that mathematics is in some way independent of, and prior to, experiential and intuitive knowledge. These views can be seen in the works of Nicholas of Cusa, Johann Kepler, Galileo, and to some extent, Leonardo da Vinci.

This idea that mathematics is the basis of the architecture of the universe was again modified in the 16th and 17th centuries. In mathematics the change came about from the use of practical use of algebra. In natural science the change was due to the rise of the experimental method. Therefore when Descartes, Boyle or others spoke about the "certainty" of mathematics it was to be interpreted to mean a consistency in its reasoning that one could find rather than in any prior necessity of existence.

In the eighteenth century we see the attention being placed on the procedures rather than the bases of mathematics through the great success in applying calculus to scientific and mathematical problems. Beginning with Leibniz, the focus shifted strongly to the relationship between mathematics and logic. This view dominated the philosophy of mathematics through the time of Frege and of Russell, but was brought into question by developments in the late 19th and early 20th century.

This leads us to the area of *logicism*, which is the thesis that mathematics is reducible to logic, and hence nothing but a part of logic. Logicists hold that mathematics can be known *a priori*, but suggest that our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. In this view, logic is the proper foundation of mathematics, and all mathematical statements are necessary logical truths. Rudolf Carnap (1931) presents the logicist thesis in two parts:

1. The concepts of mathematics can be derived from logical concepts through explicit definitions.

2. The theorems of mathematics can be derived from logical axioms through purely logical deduction.

Gottlob Frege was the founder of logicism. In his seminal *Die Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*) he built up arithmetic from a system of logic with a general principle of comprehension, which he called "Basic Law V" (for concepts F and G, the extension of F equals the extension of G if and only if for all objects a, Fa if and only if Ga), a principle that he took to be acceptable as part of logic.

But Frege's construction was flawed. Russell discovered that Basic Law V is inconsistent (this is Russell's paradox). Frege abandoned his logicist program soon after this, but it was continued by Russell and Whitehead. They attributed the paradox to "vicious circularity" and built up what they called ramified type theory to deal with it. In this system, they were eventually able to build up much of modern mathematics but in an altered, and excessively complex, form (for example, there were different natural numbers in each type, and there were infinitely many types). They also had to make several compromises in order to develop so much of mathematics, such as an "axiom of reducibility". Even Russell said that this axiom did not really belong to logic. Modern logicists have returned to a program closer to Frege's. They have abandoned Basic Law V in favor of abstraction principles such as Hume's principle. If mathematics is a part of logic, then questions about mathematical objects reduce to questions about logical objects. But what, one might ask, are the objects of logical concepts? In this sense, logicism can be seen as shifting questions about the philosophy of mathematics to questions about logic without fully answering them.

To respond to this there is the philosophy of *empiricism*. This is a form of realism that denies that mathematics can be known *a priori* at all. It says that we discover mathematical facts by empirical research, just like facts in any of the other sciences.

This view primarily arose in the middle of the 20th century. An important early proponent of a view like this was John Stuart Mill. Mill's view was widely criticized, because it makes statements like "2 + 2 = 4" come out as uncertain, contingent truths, which we can only learn by observing instances of two pairs coming together and forming a quartet. Contemporary mathematical empiricism, formulated by Quine and Putnam, is primarily supported by the indispensability argument: mathematics is indispensable to all empirical sciences, and if we want to believe in the reality of the phenomena described by the sciences, we ought to also believe in the reality of those entities required for this description. That is, since physics needs to talk about electrons to say why light bulbs behave as they do, then electrons must exist. Since physics needs to talk about numbers in offering any of its explanations, then numbers must exist. In keeping with Quine and Putnam's overall philosophies, this is a naturalistic argument. It argues for the existence of mathematical entities as the best explanation for experience, thus stripping mathematics of some of its distinctness from the other sciences.

The most important criticism of empirical views of mathematics is approximately the same as that raised against Mill. If mathematics is just as empirical as the other sciences, then this suggests that its results are just as fallible as theirs, and just as contingent. In Mill's case the empirical justification comes directly, while in Quine's case it comes indirectly, through the coherence of our scientific theory as a whole. Quine suggests that mathematics seems completely certain because the role it plays in our web of belief is incredibly central, and that it would be extremely difficult for us to revise it, though not impossible.

In order to try to address the issues raised by Russell and Whitehead that "all is logic" at the turn of the 20th century. David Hilbert introduced the view known as formalism. Formalism holds that mathematical statements may be thought of as statements about the consequences of certain string manipulation rules. For example, in the "game" of Euclidean geometry (which is seen as consisting of some strings called "axioms", and some "rules of inference" to generate new strings from given ones), one can prove that the Pythagorean theorem holds (that is, you can generate the string corresponding to the Pythagorean theorem). Mathematical truths are not about numbers and sets and triangles and the like - in fact, they aren't "about" anything at all! Another version of formalism is often known as deductivism. In deductivism, the Pythagorean theorem is not an absolute truth, but a relative one: if you assign meaning to the strings in such a way that the rules of the game become true (*i.e.*, true statements are assigned to the axioms and the rules of inference are truth-preserving), then you have to accept the theorem, or, rather, the interpretation you have given it must be a true statement. The same is held to be true for all other mathematical statements. Thus, formalism need not mean that mathematics is nothing more than a meaningless symbolic game. It is usually hoped that there exists some interpretation in which the rules of the game hold. But it does allow the working mathematician to continue in his or her work and leave such problems to the philosopher or scientist.

Many formalists would say that in practice, the axiom systems to be studied will be suggested by the demands of science or other areas of mathematics.

Hilbert's program was a complete and consistent axiomatization of all of mathematics. ("Consistent" here means that no contradictions can be derived from the system.) Hilbert aimed to show the consistency of mathematical systems from the assumption that the "finitary arithmetic" (a subsystem of the usual arithmetic of the positive integers, chosen to be philosophically uncontroversial) was consistent. Hilbert's goals of creating a system of mathematics that is both complete and consistent was dealt a fatal blow by the second of Gödel's incompleteness theorems, which states that sufficiently expressive consistent axiom systems can never prove their own consistency. Since any such axiom system would contain the finitary arithmetic as a subsystem, Gödel's theorem implied that it would be impossible to prove the system's consistency relative to that (since it would then prove its own consistency, which Gödel had shown was impossible). Thus, in order to show that any axiomatic system of mathematics is in fact consistent, one needs to first assume the consistency of a system of mathematics that is in a sense stronger than the system to be proven consistent.

Hilbert was initially a deductivist, but, as may be clear from above, he considered certain metamathematical methods to yield intrinsically meaningful results and was a realist with respect to the finitary arithmetic. Later, he held the opinion that there was no other meaningful mathematics whatsoever, regardless of interpretation. The main critique of formalism is that the actual mathematical ideas that occupy mathematicians are far removed from the minute string manipulation games mentioned above. While published proofs (if correct) could in principle be formulated in terms of these games, the effort required in space and time would be prohibitive (witness *Principia Mathematica*). In addition, the rules are certainly not substantial to the initial creation of those proofs. Formalism is also silent to the question of which axiom systems ought to be studied.

L.E.J. Brouwer developed the school of *intuitionism*. In mathematics, intuitionism is a program of methodological reform whose motto is that "there are no non-experienced mathematical truths." From this springboard, intuitionists seek to reconstruct what they consider to be the corrigible portion of mathematics in accordance with Kantian concepts of being, becoming, intuition, and knowledge. Brouwer held that mathematical objects arise from the *a priori* forms of the volitions that inform the perception of empirical objects. Leopold Kronecker said: "The natural numbers come from God, everything else is man's work."

Brouwer rejected the usefulness of formalized logic of any sort for mathematics. His student Arend Heyting, postulated an intuitionistic logic, different from the classical Aristotelian logic; this logic does not contain the law of the excluded middle and therefore frowns upon proofs by contradiction. The axiom of choice is also rejected in most intuitionistic set theories, though in some versions it is accepted. In intuitionism, the term "explicit construction" is not cleanly defined, and that has led to criticisms.

Like intuitionism, constructivism involves the regulative principle that only mathematical entities which can be explicitly constructed in a certain sense should be admitted to mathematical discourse. In this view, mathematics is an exercise of the human intuition, not a game played with meaningless symbols. Instead, it is about entities that we can create directly through mental activity. In addition, some adherents of these schools reject non-constructive proofs, such as a proof by contradiction.

Where does this leave us? Is mathematics purely a logical game played with strings that have an arbitrary truth value? If so, then why does it have such utility? Does that imply that mathematics is purely empirical? If so, are there any mathematical truths or are all statements relative? Does that make every homework paper correct?

How would you answer the following? If (or when, depending on your X-files rating) we meet an sentient being from another world, will mathematics be the true language by which we can communicate? Will their mathematics be the same as ours?