Chapter 2

NUMB3RS

2.1 The Natural Numbers $\mathbb{N}$

We should begin a discussion of real numbers by looking at the concepts of magnitude and number in ancient Greek times. The first of these might refer to the length of a geometrical line while the second concept, namely number, was thought of as composed of units. Pythagoras seems to have thought that “All is number;” so what was a number to Pythagoras?

It seems clear that Pythagoras would have thought of 1, 2, 3, 4, ... (the natural numbers in the terminology of today) in a geometrical way, not as lengths of a line as we do, but rather in the form of discrete points.

Addition, subtraction and multiplication of integers are natural concepts with this type of representation but there seems to have been no notion of division. A mathematician of this period, given the number 12, could easily see that 4 is a submultiple of it since 3 times 4 is exactly 12. Although to us this is clearly the same as division, it is important to see the distinction.

We have used the word ”submultiple”, so should indicate what the Pythagoreans considered this to be. Nicomachus, following the tradition of Pythagoras, makes the following definition of a submultiple.

Definition 2.1 *The submultiple, which is by its nature the smaller, is the number which when compared with the greater can measure it more times than one so as to fill it out exactly.*

Magnitudes, being distinct entities from numbers, had to have a separate definition and indeed Nicomachus makes such a parallel definition for magnitudes.

The idea of Pythagoras that “all is number” is explained by Aristotle in *Metaphysics*

> [*In the time of Pythagoras*] *since all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they fitted into their scheme ... the Pythagoreans say that things are what they are by intimating numbers ... the Pythagoreans take objects themselves to be numbers and do not treat mathematical objects as distinct from them ...*

Of course the Greeks were not the first to use numbers or numerals. The history points to the use of tally sticks or unary based numbers as early as 30,000 BC. The first major advance in abstraction was the use of numerals to represent numbers. This allowed systems to be developed for recording large numbers.

2.1.1 Early Chinese Mathematics

Our knowledge of early Chinese mathematics is very sketchy. There have been recent discoveries that take us back to about 450 BC. What is known is that the mathematics is very pragmatic and concise. There is a concept of proof, but it differs from that of the Western mathematics. The number system was a traditional decimal notation with one symbol for each of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000, and 10000. For example, 2048 would be written with symbols for 2,1000,4,10,8, meaning $(2 \times 1000) + (4 \times 10) + 8$. Their calculations were performed using small bamboo counting rods. The positions of the rods gave a decimal place-value system, and these were also written for long-term records. The 0 digit was a space. These numerals were arranged left to right like Arabic numerals.

In order to do addition the counting rods for the two numbers were placed down, one number above the other. The digits were added (merged) left to right with carries where needed. Subtraction was done similarly. The multiplication table to 9 times 9 were memorized. A long multiplication was similar to ours but with with advantages due to physical rods. Long division was analogous to current algorithms, but closer to “galley method,” or “scratch method” said to have been developed in India.

2.1.2 Early Indian Mathematics

Our knowledge of Indian mathematics is less sketchy, but also less studied in Western thought. The first appearance of evidence of the use of mathematics in the Indian subcontinent was in the Indus Valley Civilization, which dates back to around 3300

BC. Excavations at Harappa, Mohenjo-daro and the surrounding area of the Indus River, have uncovered much evidence of the use of basic mathematics. The mathematics used by this early Harappan civilization was very much for practical means, and was primarily concerned with:

- weights and measuring scales;
- an advanced brick technology, which used ratios.

The achievements of the Harappan people of the Indus Valley Civilization include:

- Great accuracy in measuring length, mass, and time.
- The first system of uniform weights and measures.
- Extremely precise measurements.[1]
- Also of great interest is a remarkably accurate decimal ruler known as the Mohenjo-daro ruler. Subdivisions on the ruler have a maximum error of just 0.005 inches and, at a length of 1.32 inches, have been named the Indus inch.
- Some historians believe the Harappan civilization may have used a base 8 numeral system.

Later contributions from Vedic mathematics (1500 BC to 400 BC) and Jaina mathematics (400 BC to 200 BC) include the use of zero, negative numbers, and the Hindu-Arabic numeral system in use today. The golden age of Indian mathematics is said to have been 400 to 1200 AD in which we find the discoveries of sine, cosine, solutions to quadratic equations, exponentials, logarithms, summing series, and much more which was lost to Western mathematics.

2.1.3 Babylonian numerals

The Babylonians developed a powerful place-value system based essentially on the numerals for 1 and 10. Their number system was a sexagesimal (base 60) positional number system inherited from the Sumerian and also Akkadian civilizations. Babylonian numerals were written in cuneiform, using a wedge-tipped reed stylus to make a mark on a soft clay tablet which would be exposed in the sun to harden to create a permanent record.

This system first appeared around 1900 BC to 1800 BC. It is also credited as being the first known place-value numeral system, in which the value of a particular digit depends both on the digit itself and its position within the number. This was an extremely important development, because prior to place-value systems people were

[1] Their smallest division, which is marked on an ivory scale found in Lothal, was approximately 1.704mm, the smallest division ever recorded on a scale of the Bronze Age.

obliged to use unique symbols to represent each power of a base (ten, one-hundred, one thousand, and so forth), making even basic calculations unwieldy.

Sexagesimals still survive to this day, in the form of degrees (360 in a circle), minutes, and seconds in trigonometry and the measurement of time.

A common theory is that sixty was chosen due to its prime factorization 2235 which makes it divisible by 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. Integers and fractions were represented identically - a radix(decimal) point was not written but rather made clear by context.

More is available in Appendix A.

2.1.4 Egyptian numerals

The ancient Egyptians had a system of numerals with distinct hieroglyphs for 1, 10, and all the powers of 10 up to one million. A stone carving from Karnak, dating from around 1500 BC and now at the Louvre in Paris, depicts 276 as 2 hundreds, 7 tens, and 6 ones; and similarly for the number 4,622.

Circa 3100 BC Egyptians introduced the earliest known decimal system, allowing the use of large numbers and also fractions in the form of unit fractions and Eye of Horus fractions. Clear records began to appear by 2,000 BC citing approximations for π and square roots. Exact statements of number, written arithmetic tables, algebra problems, and practical applications with weights and measures also began to appear around 2,000 BC.

Two number systems were used in ancient Egypt. One, written in hieroglyphs, was a decimal based tally system with separate symbols for 10, 100, 1000, etc, as Roman numerals were later written, and hieratic unit fractions. The hieroglyphic number system existed from at least the Early Dynastic Period.

The hieratic system differed from the hieroglyphic system beyond a use of simplifying ligatures for rapid writing and began around 2150 BC. Boyer proved 50 years ago that hieratic script used a different numeral system, using individual signs for the numbers 1 to 9, multiples of 10 from 10 to 90, the hundreds from 100 to 900, and the thousands from 1000 to 9000. A large number like 9999 could thus be written with only four signscombining the signs for 9000, 900, 90, and 9 – as opposed to 36 hieroglyphs. Boyer saw the new hieratic numerals as ciphered, mapping one number onto one Egyptian letter for the first time in Western history. Greeks adopted the new system, mapping their counting numbers onto two of their alphabets, the Doric and Ionian.

Egyptian addition was done by tallying glyphs and subtraction was done similarly. How is that technique being used today?

Egyptian multiplication was based on the process of doubling and halving. It has been modified and that modification is called the Russian Peasant Method of multiplication. Division can be done similarly, but there are more complications when you can only represent fractions as unit fractions.

Egyptian fractions were of only one kind — fractions with one as a numerator with the exception of 2/3 and 3/4. Every other fraction had to be written as a sum of fractions of the form $1/n$.

2.2 Zero

A much later advance in abstraction was the development of the idea of zero as a number with its own numeral. A zero digit had been used in place-value notation as early as 700 BC by the Babylonians, but it was never used as a final element. The Olmec and Maya civilization used zero as a separate number as early as 4th century BC but this usage did not spread beyond Mesoamerica. The concept as used in modern times originated with the Indian mathematician Brahmagupta in 628. Nevertheless, zero was used as a number by all medieval computists (calculators of Easter) beginning with Dionysius Exiguus in 525, but in general no Roman numeral was used to write it. Instead, the Latin word for "nothing," *nullæ*, was employed.

The use of zero as a number should be distinguished from its use as a placeholder numeral in place-value systems. Many ancient Indian texts use a Sanskrit word Shunya to refer to the concept of void; in mathematics texts this word would often be used to refer to the number zero.

Records show that the Ancient Greeks seemed unsure about the status of zero as a number: they asked themselves "how can 'nothing' be something?", leading to interesting philosophical and, by the Medieval period, religious arguments about the nature and existence of zero and the vacuum. The paradoxes of Zeno of Elea depend in large part on the uncertain interpretation of zero.

The late Olmec people of south-central Mexico began to use a true zero (a shell glyph) in the New World possibly by the 4th century BC but certainly by 40 BC, which became an integral part of Maya numerals and the Maya calendar, but did not influence Old World numeral systems.

By 130 CE Ptolemy, influenced by Hipparchus and the Babylonians, was using a symbol for zero (a small circle with a long overbar) within a sexagesimal numeral system otherwise using alphabetic Greek numerals. Because it was used alone, not as just a placeholder, this Hellenistic zero was the first documented use of a true zero in the Old World. In later Byzantine manuscripts of his Syntaxis Mathematica (Almagest), the Hellenistic zero had morphed into the Greek letter omicron (otherwise meaning 70).

Another true zero was used in tables alongside Roman numerals by 525 CE (first known use by Dionysius Exiguus), but as a word, *nulla* meaning nothing, not as a symbol. When division produced zero as a remainder, *nihil*, also meaning nothing, was used. These medieval zeros were used by all future medieval computists. An isolated use of the initial, N, was used in a table of Roman numerals by Bede or a colleague about 725 CE, a true zero symbol.

An early documented use of the zero by Brahmagupta (in the *Brahmasphutasiddhanta*) dates to 628 CE. He treated zero as a number and discussed operations involving it, including division. By this time (7th century) the concept had clearly reached Cambodia, and documentation shows the idea later spreading to China and the Islamic world.

2.3 Negative Numbers

For a long time, negative solutions to problems were considered "false" because they couldn't be found in the real world (in the sense that one cannot have a negative number of material objects). The abstract concept was recognized as early as 100 BC - 50 BC. The Chinese *Nine Chapters on the Mathematical Art* (Jiu-zhang Suanshu) contains methods for finding the areas of figures; red rods were used to denote positive coefficients, black for negative. They were able to solve simultaneous equations involving negative numbers. At around the same time in ancient India, the Bakhshali manuscript written sometime between 200 BC and 200 CE carried out calculations with negative numbers, using a + as a negative sign. These are the earliest known uses of negative numbers.

In Hellenistic Egypt, Diophantus in the 3rd century CE referred to the equation equivalent to $4x+20=0$ (the solution would be negative) in *Arithmetica*, saying that the equation was absurd, indicating that no concept of negative numbers existed in the ancient Mediterranean.

During the 7th century, negative numbers were in use in India to represent debts. The Indian mathematician Brahmagupta, in *Brahma-Sphuta-Siddhanta* (written in 628 CE) discusses the use of negative numbers to produce the general form quadratic formula that remains in use today. He also finds negative solutions to quadratic equations and gives rules regarding operations involving negative numbers and zero, such as "a debt cut off from nothingness becomes a credit, a credit cut off from nothingness becomes a debt." He called positive numbers "fortunes," zero a "cipher," and negative numbers a "debt". In the 12th century in India, Bhaskara also gives negative roots for quadratic equations but rejects the negative roots since they were inappropriate in the context of the problem, stating that the negative values "is in this case not to be taken, for it is inadequate; people do not approve of negative roots."

From the 8th century, the Islamic world learnt about negative numbers from Arabic translations of Brahmagupta's works, and by about 1000 CE, Arab mathematicians had realized the use of negative numbers for debt.

Knowledge of negative numbers eventually reached Europe through Latin translations of Arabic and Indian works.

European mathematicians however, for the most part, resisted the concept of negative numbers until the 17th century, although Fibonacci allowed negative solutions in financial problems where they could be interpreted as debits (chapter 13 of *Liber*

Abaci, 1202) and later as losses (in *Flos*). At the same time, the Chinese were indicating negative numbers by drawing a diagonal stroke through the right-most non-zero digit. The first use of negative numbers in a European work was by Chuquet during the 15th century. He used them as exponents, but referred to them as "absurd numbers".

The English mathematician Francis Maseres wrote in 1759 that negative numbers "darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple." He came to the conclusion that negative numbers did not exist.

Negative numbers were not well-understood until modern times. As recently as the 18th century, the Swiss mathematician Leonhard Euler believed that negative numbers were greater than infinity, and it was common practice to ignore any negative results returned by equations on the assumption that they were meaningless.

2.4 The Integers $\mathbb{Z}$

The collection of natural numbers, zero and the negative numbers form the set we call the integers, denoted by $\mathbb{Z}$. This is the set with which we have the most experience. We feel that we "understand" the integers. We know how to add and multiply and subtract them. We know the properties of 0 and we know how to deal with the addition, subtraction and multiplication of negative numbers.

The integers do not have one important property - closure under division, or to state this in mathematical terms, not every non-zero integer has a multiplicative inverse. In some sense we would "like" the integers to satisfy similar properties for addition and multiplication. This is the desire for "aesthetic beauty" in mathematics. It just "looks right."

One important group of integers that we will use is the collection of primes - numbers that are divisible only by themselves and the unit, 1. These have generated many problems and results in the history of mathematics.

Theorem 2.1 (Fundamental Theorem of Arithmetic) *Let $n \in \mathbb{Z}^+$ be a positive integer, then there is a unique representation of the form*

$$n = {p_1}^{n_1} {p_2}^{n_2} \dots {p_m}^{n_m},$$

where all of the p_i's are prime and the n_i's are positive.

2.5 The Rational Numbers $\mathbb{Q}$

We have seen that the Egyptians had considered reciprocals and certain fractions, as had the Babylonians. Information suggests that the Chinese mathematicians and

Indian mathematicians also understood and worked with fractions and had ways to add, multiply, subtract and divide fractions.

This is just to say that the idea of a rational number was around well before the Greeks. The Greeks however considered fractions as ratios, not numbers in the sense of whole numbers.

In Book V Euclid considers magnitudes and the theory of proportion of magnitudes. This was, most likely, the work of Eudoxus, and this is stated in a later version of *The Elements*. When Euclid illustrates a theorem about magnitudes he usually gives a diagram representing the magnitude by a line segment. Magnitude, however, is an abstract concept to Euclid so it applies to lines, surfaces and solids and Euclid knows that his theory applies to time and angles.

Given the emphasis placed on Euclid and *The Elements* and his axiomatic approach to mathematics, we would expect him to begin with a definition of magnitude and state some unproved axioms. He chooses to leave the concept of ***magnitude*** undefined and his first two definitions refer to the *part* of a magnitude and a *multiple* of a magnitude:

Definition 2.2 (Definition V.1) *A magnitude is a* ***part*** *of a magnitude, the less of the greater, when it measures the greater.*

Euclid leaves the term *measures* undefined but it seems that Euclid means that (in modern symbols) the smaller magnitude x is a part of the greater magnitude y if $nx = y$ for some natural number $n > 1$.

Definition 2.3 (Definition V.2) *The greater is a* ***multiple*** *of the less when it is measured by the less.*

Euclid then defines *ratio*.

Definition 2.4 (Definition V.3) *A* ***ratio*** *is a sort of relation in respect of size between two magnitudes of the same kind.*

This is not unlike some of his other definitions in that it fails to define ***ratio*** at all. Euclid then defines when magnitudes have a ratio. According to his definition this is when there is an multiple (by a natural number) of the first which exceeds the second and a multiple of the second which exceeds the first.

Then comes the vital definition of when two magnitudes are in the same ratio as a second pair of magnitudes. It can be difficult to fathom Euclid's language, so we shall translate it into modern notation.

It says that $a : b = c : d$ if given any natural numbers n and m we have

$na > mb$ if and only if $nc > md$
$na = mb$ if and only if $nc = md$
$na < mb$ if and only if $nc < md$.

In Book VII Euclid studies numbers and some number theory. He defines a unit, then a number is defined as being composed of a multitude of units, and parts and multiples are defined as for magnitudes. Recall that Euclid, as earlier Greek mathematicians, did not consider 1 as a number. It was a unit and the numbers $2, 3, 4, \ldots$ were composed of units.

As is true in earlier books, various properties of numbers are assumed but are not listed as axioms. For example the commutative law for multiplication is assumed without ever being stated as an axiom as are the associative law for addition etc.

Euclid then introduces *proportion* for numbers and shows that for numbers a, b, c, d it is true that $a : b = c : d$ precisely when the least numbers with ratio $a : b$ are equal to the least numbers with ratio $c : d$. This is logically equivalent to saying in modern terms that the rational a/b and the rational c/d are equal if they become the same when reduced to their lowest terms.

An important result in Book VII is the Euclidean algorithm. Note that Euclid never identified the ratio $2 : 1$ with the number 2. To the Greeks these were two quite different concepts.

Book X considers *commensurable* and *incommensurable* magnitudes. It is a long book, over one quarter of the whole of *The Elements*. In Book X we find

Definition 2.5 (Definition X.1) *Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.*

Euclid then proves several results, among them:

Proposition 2.1 (Proposition X.2) *If, when two unequal magnitudes are set out and the lesser is always subtracted in turn from the greater, the remainder never measures the magnitude before it, then the magnitudes will be incommensurable.*[2]

Proposition 2.2 (Proposition X.5) *Commensurable magnitudes have to one another the ratio which a number has to a number.*

Euclid also proves some of the results of Theodorus, namely that segments of length $\sqrt{3}, \sqrt{5}, \ldots, \sqrt{17}$ are incommensurable with a segment of unit length.

This takes us away from the rational numbers though. A major advance was made by Stevin in 1585 in *La Theinde* when he introduced decimal fractions. One has to understand here that in fact it was in a sense fortuitous that his invention led to a much deeper understanding of numbers for he certainly did not introduce the notation with that in mind. Only finite decimals were allowed, so with his notation only certain rationals to be represented exactly. Other rationals could be represented approximately and Stevin saw the system as a means to calculate with approximate

[2]This says that two magnitudes are incommensurable if the Euclidean algorithm does not terminate.

rational values. His notation was to be taken up by Clavius and Napier but others resisted using it since they saw it as a backwards step to adopt a system which could not even represent 1/3 exactly.

2.6 The Irrational Numbers

So to the ancient Greeks numbers were $1, 2, 3, \ldots$ and ratios of numbers were used which (although not considered to be numbers) basically allowed manipulation with what we call rationals. Also magnitudes were considered and these were essentially lengths constructible by ruler and compass from a line of unit length. No other magnitudes were considered. Hence mathematicians studied magnitudes which had lengths which, in modern terms, could be formed from positive integers by addition, subtraction, multiplication, division and taking square roots.

The Arabic mathematicians went further with constructible magnitudes for they used geometric methods to solve cubic equations which meant that they could construct magnitudes whose ratio to a unit length involved cube roots. For example Omar Khayyam showed how to solve all cubic equations by geometric methods. Fibonacci, using skills learnt from the Arabs, solved a cubic equation showing that its root was not formed from rationals and square roots of rationals as Euclid's magnitudes were. He then went on to compute an approximate solution.

Although no conceptual advances were taking place, by the end of the fifteenth century mathematicians were considering expressions built from positive integers by addition, subtraction, multiplication, division and taking nth roots. These are called *radical expressions*. By the sixteenth century rational numbers and roots of numbers were becoming accepted as numbers although there was still a sharp distinction between these different types of numbers. Stifel, in his *Arithmetica Integra* (1544) argues that irrationals must be considered valid

> *It is rightly disputed whether irrational numbers are true numbers or false. Because in studying geometrical figures, where rational numbers desert us, irrationals take their place, and show precisely what rational numbers are unable to show ... we are moved and compelled to admit that they are correct ...*

He continues to argue that, as they are not proportional to rational numbers, they cannot be true numbers even if they are correct. He ends up arguing that all irrational numbers result from radical expressions.

One obvious question we want to ask Stifel is: what about the length of the circumference of a circle with radius of unit length? In fact Stifel gives an answer to this in an appendix to the book. First he makes a distinction between physical circles and mathematical circles. One can measure the properties of physical circles, he claims, but one cannot measure a mathematical circle with physical instruments.

He then goes on to consider the circle as the limit of a sequence of polygons of more and more sides. He writes:- Therefore the mathematical circle is rightly described as the polygon of infinitely many sides. And thus the circumference of the mathematical circle receives no number, neither rational nor irrational. Not too good an argument, but nevertheless a remarkable insight that there were lengths which did not correspond to radical expressions but which could be approximated as closely as one wished.

Stevin made a number of other important advances in the study of the real numbers. He argued strongly in *L'Arithmetique* (1585) that all numbers such as square roots, irrational numbers, surds, negative numbers *etc.* should all be treated as numbers and not distinguished as being different in nature. He wrote

> *Thesis 1: That unity is a number.*
> *Thesis 2: That any given numbers can be square, cubes, fourth powers etc.*
> *Thesis 3: That any given root is a number.*
> *Thesis 4: That there are no absurd, irrational, irregular, inexplicable or surd numbers.*
> *It is a very common thing amongst authors of arithmetics to treat numbers like* $\sqrt{8}$ *and similar ones, which they call absurd, irrational, irregular, inexplicable or surds etc and which we deny to be the case for number which turns up.*

His first thesis was to argue against the Greek idea that 1 is not a number but a unit and the numbers $2, 3, 4, \ldots$ were composed of units. The other three theses were encouraging people to treat different types of numbers, which were at that time treated separately, as a single entity — namely a number.

One further comment by Stevin in *L'Arithmetique* is worth recording. He noted that Euclid's Proposition X.2 says that two magnitudes are incommensurable if the Euclidean algorithm does not terminate. Stevin writes about this pointing out in today's language the difference between an algorithm and a procedure (or semi-algorithm)

> *Although this theorem is valid, nevertheless we cannot recognise by such experience the incommensurability of two given magnitudes. ... even though it were possible for us to subtract by due process several hundred thousand times the smaller magnitude from the larger and continue that for several thousands of years, nevertheless if the two given numbers were incommensurable one would labour eternally, always ignorant of what could still happen in the end. This manner of cognition is therefore not legitimate, but rather an impossible position ...*

The irrationality of e was proven by Euler in 1737. π^n is irrational for positive integers n. The irrationality of π itself was proven by Lambert in 1760. The Erdös-

Borwein constant

$$\begin{aligned} E &= \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \\ &\approx 1.606695152415291763\ldots \end{aligned}$$

is known to be irrational.

From Gelfond's theorem [2.2], a number of the form a^b is transcendental (and therefore irrational) if a is algebraic $a \neq 0, 1$ and b is irrational and algebraic. This establishes the irrationality of Gelfond's constant e^π and $2^{\sqrt{2}}$. Nesterenko (1996) proved that $\pi + e^\pi$ is irrational.

It is not known whether $\pi + e$ and $\pi - e$ are irrational or not. In fact, there is no pair of non-zero integers m and n for which it is known whether $m\pi + ne$ is irrational or not. Irrationality has not yet been established for 2^e, π^e, $\pi^{\sqrt{2}}$, or γ, where

$$\begin{aligned} \gamma &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \\ &= -\int_0^\infty e^{-x} \ln x dx \\ &= -\int_0^1 \ln\ln\left(\frac{1}{x}\right) dx \\ &\approx 0.57721566490153286065120900824024310 42\ldots \end{aligned}$$

is the Euler-Mascheroni constant[3].

2.7 The Real Numbers $\mathbb{R}$

By the time Stevin proposed the use of decimal fractions in 1585, the concept of a number had developed little from that of Euclid's *Elements*. If we move forward almost exactly 100 years to the publication of *A treatise of Algebra* in 1684 Wallis accepts, without any great enthusiasm, the use of Stevin's decimals. He still only considers finite decimal expansions and realizes that with these one can approximate numbers (which for him are constructed from positive integers by addition, subtraction, multiplication, division and taking nth roots) as closely as one wishes. However, Wallis understood that there were proportions which did not fall within this definition of number, such as those associated with the area and circumference of a circle

[3]The English mathematician G.H. Hardy said to have stated that he would give up his endowed chair at Oxford to anyone who could prove γ to be irrational.

> *...such proportion is not to be expressed in the commonly received ways of notation: particularly that for the circles quadrature. ...Now, as for other incommensurable quantities, though this proportion cannot be accurately expressed in absolute numbers, yet by continued approximation it may; so as to approach nearer to it than any difference assignable.*

For Wallis there were a number of ways to achieve this approximation, coming as close as needed. He considered approximations by continued fractions, and also approximations by taking successive square roots. This leads into the study of infinite series. Without the necessary machinery to prove that these infinite series converged to a limit, however, he was never going to be able to progress much further in studying real numbers. Real numbers became very much associated with magnitudes. No definition was really thought necessary, and in fact the mathematics was considered the science of magnitudes. Euler, in *Complete introduction to algebra* (1771) wrote in the introduction

> *Mathematics, in general, is the science of quantity; or, the science which investigates the means of measuring quantity.*

Euler also defined the notion of quantity as "that which can be continuously increased or diminished" and considered length, area, volume, mass, velocity, time, *etc.* to be different examples of quantity. All could be measured by real numbers. However, Euler's mathematics itself led to a more abstract idea of quantity — a variable x which need not necessarily take real values. Symbolic mathematics took the notion of quantity too far, and a reassessment of the concept of a real number became more necessary. By the beginning of the nineteenth century a more rigorous approach to mathematics, principally by Cauchy and Bolzano, began to provide the machinery to put the real numbers on a firmer footing.

Cauchy, in *Cours d'analyse* (1821), did not worry too much about the definition of the real numbers. He does say that a *real number* is the limit of a sequence of rational numbers but he is assuming here that the real numbers are known. Certainly this is not considered by Cauchy to be a definition of a real number, rather it is simply a statement of what he considers an "obvious" property. He says nothing about the need for the sequence to be what we call today a *Cauchy sequence* and this is necessary if one is to define convergence of a sequence without assuming the existence of its limit. He does define the product of a rational number A and an irrational number B as follows:-

> *Let b, b', b'', ... be a sequence of rationals approaching B closer and closer. Then the product AB will be the limit of the sequence of rational numbers Ab, Ab', Ab'', ...*

Bolzano, on the other hand, showed that bounded Cauchy sequence of real numbers had a least upper bound in 1817. He later worked out his own theory of real numbers which he did not publish. This was a quite remarkable achievement and it is only comparatively recently that we have understood exactly what he did achieve. His definition of a real number was made in terms of convergent sequences of rational numbers. In J. Berg's edition of Bolzano's *Reine Zahlenlehre* which was published in 1976, Berg points out that Bolzano had discovered some difficulties in his work himself, and Berg found notes by Bolzano which proposed amendments to his theory which make it completely correct. As Bolzano's contributions were unpublished they had little influence in the development of the theory of the real numbers.

Cauchy himself does not seem to have understood the significance of his own "Cauchy sequence" criterion for defining the real numbers. Nor did his immediate successors. It was Weierstrass, Heine, Mray, Cantor and Dedekind who, after convergence and uniform convergence were better understood, were able to give rigorous definitions of the real numbers.

Up to this time there was no proof that numbers existed that were not the roots of polynomial equations with rational coefficients. Clearly $\sqrt{2}$ is the root of a polynomial equation with rational coefficients, namely $x^2 = 2$, and it is easy to see that all roots of rational numbers arise as solutions of such equations. A number is called *transcendental* if it is not the root of a polynomial equation with rational coefficients. The word transcendental is used as such number transcend the usual operations of arithmetic. Although mathematicians had guessed for a long time that π and e were transcendental, this had not been proved up to the middle of the 19th century. Liouville's interest in transcendental numbers stemmed from reading a correspondence between Goldbach and Daniel Bernoulli. Liouville certainly aimed to prove that e is transcendental but he did not succeed. However his contributions led him to prove the existence of a transcendental number in 1844 when he constructed an infinite class of such numbers using continued fractions. These were the first numbers to be proved transcendental. In 1851 he published results on transcendental numbers removing the dependence on continued fractions. In particular he gave an example of a transcendental number, the number now named the Liouvillian number

$$0.1100010000000000000000010000...$$

where there is a 1 in place $n!$ and 0 elsewhere.

One of the first people to attempt to give a rigorous definition of the real numbers was Hamilton. Hamilton came close to the idea of a *Dedekind cut*, as Euclid had done in *The Elements*, but failed to make the idea into a definition (again Euclid had spotted the property but never thought to use it as a definition). For a number a he noted that there are rationals a', a'', b', b'', c', c'', d', d'', ... with

$$\begin{array}{l} a' < a < a'' \\ b' < a < b'' \\ c' < a < c'' \\ d' < a < d'' \\ etc. \end{array}$$

but he never thought to define a number by the sets $\{a', b', c', d', \dots\}$ and $\{a", b", c", d", \dots\}$. He tried another approach of defining numbers given by some law, say $x \mapsto x^2$. Hamilton writes

> *If x undergoes a continuous and constant increase from zero, then will pass successively through every state of positive ration b, and therefore that every determined positive ration b has one determined square root $\sqrt{b}$ which will be commensurable or incommensurable according as b can or cannot be expressed as the square of a fraction. When b cannot be so expressed, it is still possible to approximate in fractions to the incommensurable square root $\sqrt{b}$ by choosing successively larger and larger positive denominators ...*

One can see what Hamilton is getting at, but much here is without justification. Can a quantity undergo a continuous and constant increase? Even if one got round this problem he is only defining numbers given by a law. It is unclear whether he thought that all real numbers would arise in this way.

When progress came in giving a rigorous definition of a real number, there was a sudden flood of contributions. Dedekind worked out his theory of Dedekind cuts in 1858 but it remained unpublished until 1872. Weierstrass gave his own theory of real numbers in his Berlin lectures beginning in 1865 but this work was not published. The first published contribution regarding this new approach came in 1867 from Hankel who was a student of Weierstrass. Hankel, for the first time, suggests a total change in out point of view regarding the concept of a real number

> *Today number is no longer an object, a substance which exists outside the thinking subject and the objects giving rise to this substance, an independent principle, as it was for instance for the Pythagoreans. Therefore, the question of the existence of numbers can only refer to the thinking subject or to those objects of thought whose relations are represented by numbers. Strictly speaking, only that which is logically impossible (i.e. which contradicts itself) counts as impossible for the mathematician.*

In his 1867 monograph Hankel addressed the question of whether there were other "number systems" which had essentially the same rules as the real numbers.

Two years after the publication of Hankel's monograph, Mray published *Remarques sur la nature des quantités* in which he considered Cauchy sequences of rational numbers which, if they did not converge to a rational limit, had what he called a "fictitious limit." He then considered the real numbers to consist of the rational numbers and his fictitious limits. Three years later Heine published a similar notion in his book *Elemente der Functionenlehre* although it was done independently of Mray. It was similar in nature with the ideas which Weierstrass had discussed in his lectures.

Heine's system has become one of the two standard ways of defining the real numbers today. Essentially Heine looks at Cauchy sequences of rational numbers. He defines an equivalence relation on such sequences by defining

$$a_1, a_2, a_3, a_4, \ldots \text{ and } b_1, b_2, b_3, b_4, \ldots$$

to be equivalent if the sequence of rational numbers $\{a_1-b_1, a_2-b_2, a_3-b_3, a_4-b_4, \ldots\}$ converges to 0. Heine then introduced arithmetic operations on his sequences and an order relation. Particular care is needed to handle division since sequences with a non-zero limit might still have terms equal to 0.

Cantor also published his version of the real numbers in 1872 which followed a similar method to that of Heine. His numbers were Cauchy sequences of rational numbers and he used the term "determinate limit". It was clear to Hankel (see the quote above) that the new ideas of number had suddenly totally changed a concept which had been motivated by measurement and quantity. Similarly Cantor realized that if he wants the line to represent the real numbers then he has to introduce an axiom to recover the connection between the way the real numbers are now being defined and the old concept of measurement. He writes about a distance of a point from the origin on the line

> *If this distance has a rational relation to the unit of measure, then it is expressed by a rational quantity in the domain of rational numbers; otherwise, if the point is one known through a construction, it is always possible to give a sequence of rationals* $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ *which has the properties indicated and relates to the distance in question in such a way that the points on the straight line to which the distances* $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ *are assigned approach in infinity the point to be determined with increasing* n. *...In order to complete the connection presented in this section of the domains of the quantities defined* [his determinate limits] *with the geometry of the straight line, one must add an axiom which simply says that every numerical quantity also has a determined point on the straight line whose coordinate is equal to that quantity, indeed, equal in the sense in which this is explained in this section.*

As we mentioned above, Dedekind had worked out his idea of Dedekind cuts in 1858. When he realized that others like Heine and Cantor were about to publish their versions of a rigorous definition of the real numbers he decided that he too should publish his ideas. This resulted in yet another 1872 publication giving a definition of the real numbers. Dedekind considered all decompositions of the rational numbers into two sets A_1 and A_2 so that $a_1 < a_2$ for all $a_1 \in A_1$ and $a_2 \in A_2$. He called (A_1, A_2) a *cut*. If the rational a is either the maximum element of A_1 or the minimum element of A_2 then Dedekind said the cut was produced by a. However not all cuts were produced by a rational. He wrote

> *In every case in which a cut (A_1, A_2) is given that is not produced by a rational number, we create a new number, an irrational number a, which we consider to be completely defined by this cut; we will say that the number a corresponds to this cut or that it produces the cut.*

He defined the usual arithmetic operations and ordering and showed that the usual laws apply.

Another definition, similar in style to that of Heine and Cantor, appeared in a book by Thomae in 1880. Thomae had been a colleague of Heine and Cantor around the time they had been writing up their ideas. He claimed that the real numbers defined in this way had a right to exist because

> *. . . the rules of combination abstracted from calculations with integers may be applied to them without contradiction.*

Frege, however, attacked these ideas of Thomae. He wanted to develop a theory of real numbers based on a purely logical base and attacked the philosophy behind the constructions which had been published. Thomae added further explanation to his idea of "formal arithmetic" in the second edition of his text which appeared in 1898

> *The formal conception of numbers requires of itself more modest limitations than does the logical conception. It does not ask, what are and what shall the numbers be, but it asks, what does one require of numbers in arithmetic.*

Frege was still unhappy with the constructions of Weierstrass, Heine, Cantor, Thomae and Dedekind. How did one know. he asked, that the constructions led to systems which would not produced contradictions? He wrote in 1903

> *This task has never been approached seriously, let alone been solved.*

Frege, however, never completed his own version of a logical framework. His hopes were shattered when he learnt of Russell's paradox.

Hilbert had taken a totally different approach to defining the real numbers in 1900. He defined the real numbers to be a system with eighteen axioms. Sixteen of these axioms define what today we call an ordered field, while the other two were the Archimedean axiom and the completeness axiom. The Archimedean axiom stated that given positive numbers a and b then it is possible to add a to itself a finite number of times so that the sum exceed b. The completeness property says that one cannot extend the system and maintain the validity of all the other axioms. This was totally new since all other methods built the real numbers from the known rational numbers. Hilbert's numbers were unconnected with any known system. It was impossible to say whether a given mathematical object was a real number. Most seriously, there was no proof that any such system actually existed. If it did it was still subject to the same questions concerning its consistency as Frege had pointed out.

By the beginning of the 20th century, then, the concept of a real number had moved away completely from the concept of a number which had existed from the most ancient times to the beginning of the 19th century, namely its connection with measurement and quantity.

2.8 Algebraic Numbers

Up to the time of Cauchy there was no proof that numbers existed that were not the roots of polynomial equations with rational coefficients. Clearly $\sqrt{2}$ is the root of a polynomial equation with rational coefficients, namely $x^2 = 2$, and it is easy to see that all roots of rational numbers arise as solutions of such equations. A number is called *transcendental* if it is not the root of a polynomial equation with rational coefficients. The word transcendental is used as such number transcend the usual operations of arithmetic. Although mathematicians had guessed for a long time that π and e were transcendental, this had not been proved up to the middle of the 19th century. Liouville's interest in transcendental numbers stemmed from reading a correspondence between Goldbach and Daniel Bernoulli. Liouville certainly aimed to prove that e is transcendental but he did not succeed. However his contributions led him to prove the existence of a transcendental number in 1844 when he constructed an infinite class of such numbers using continued fractions. These were the first numbers to be proved transcendental. In 1851 he published results on transcendental numbers removing the dependence on continued fractions. In particular he gave an example of a transcendental number, the number now named the Liouville constant

$$L = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000 0\ldots$$

where there is a 1 in place $n!$ and 0 elsewhere.

A number is *algebraic* if it is the root of a polynomial equation with rational coefficients. These are the numbers that we tend to know. Thus, we would assume that these are numerous, while the size of transcendental numbers is small in comparison. This is not true. One can show that the set of algebraic numbers is denumerable, while the set of real numbers is non-denumerable. Since the algebraic and transcendental numbers are mutually disjoint, the set of transcendental numbers must be non-denumerable.

The number e was proven to be transcendental by Hermite in 1873, and π by Lindemann in 1882.

Gelfond's theorem, also called the Gelfond-Schneider theorem, states that

Theorem 2.2 (Gelfond-Schneider) α^β *is transcendental if*

1. *α is algebraic, $\alpha \neq 0, 1$ and*
2. *β is algebraic and irrational.*

Gelfond's constant e^{π} is transcendental by Gelfond's theorem since

$$(-1)^{-i} = (e^{i\pi})^{-i} = e^{\pi}.$$

The Gelfond-Schneider constant $2^{\sqrt{2}}$ is also transcendental as is

$$i^i = e^{-\pi/2} \approx 0.20787957635076190854695561983497877003387784163176$$

Known transcendentals include $\ln 2$, $\ln 3/\ln 2$, $\sin(a)$, $\cos(a)$ and $\tan(a)$ for any nonzero rational number a, and $\arctan(x)/\pi$ for x rational and $x \neq 0, \pm 1$.

At least one of πe and $\pi + e$ (and probably both) are transcendental, but transcendence has not been proven for either number on its own. It is **not** known if e^e, π^π, π^e, or γ (the Euler-Mascheroni constant) are transcendental.

2.9 Extensions of the Reals

2.9.1 Complex Numbers

The earliest fleeting reference to square roots of negative numbers perhaps occurred in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century CE, when he considered the volume of an impossible frustum of a pyramid, even though negative numbers were not conceived in the Hellenistic world.

Complex numbers became more prominent in the 16th century, when closed formulas for the roots of cubic and quartic polynomials were discovered by Italian mathematicians such as Niccolo Fontana Tartaglia and Gerolamo Cardano. It was soon realized that these formulas, even if one was only interested in real solutions, sometimes required the manipulation of square roots of negative numbers. For example, Tartaglia's cubic formula gives the following solution to the equation $x^3 - x = 0$:

$$\frac{1}{\sqrt{3}}\left(\sqrt{-1}^{1/3} + \frac{1}{\sqrt{-1}^{1/3}}\right).$$

Now, at first glance this looks like nonsense, since $0, \pm 1$ are the roots. Formal calculations with complex numbers, however, show that the equation $z^3 = i$ has solutions $-i$, $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ and $\dfrac{-\sqrt{3}}{2} + \dfrac{1}{2}i$. Substituting these in turn for into the cubic formula and simplifying, one gets 0, 1 and -1 as the solutions of $x^3 - x = 0$.

This was a real problem for Renaissance mathematicians since not even negative numbers were considered to be on firm ground at the time. The term *imaginary* for these quantities was coined by Ren Descartes in 1637 and ***was*** meant to be derogatory[4]. A further source of confusion was that the equation $(\sqrt{-1})^2 = \sqrt{-1}\sqrt{-1} = -1$ seemed to be capriciously inconsistent with the algebraic identity $\sqrt{ab} = \sqrt{a}\sqrt{b}$, which is valid for positive real numbers a and b, and which was also used in complex number calculations with one of a, b positive and the other negative. The incorrect use of this identity (and the related identity $\frac{1}{\sqrt{a}} = \sqrt{\frac{1}{a}}$) in the case when both a and b are negative even bedeviled Euler. This difficulty eventually led to the convention of using the special symbol i in place of $\sqrt{-1}$ to guard against this mistake.

In the 18th century Abraham de Moivre and Leonhard Euler further extended the use and utility of complex numbers. De Moivre (1730) discovered the well-known formula which bears his name, de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Euler (1748) discovered what is known as Euler's formula of complex analysis:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The existence of complex numbers was not completely accepted until an appropriate geometrical interpretation had been described by Caspar Wessel in 1799. It was rediscovered several years later and popularized by Gauss, and as a result the theory of complex numbers received a notable expansion. The idea of the graphic representation of complex numbers had appeared, however, as early as 1685, in Wallis's *De Algebra tractatus*.

Wessel's memoir appeared in the *Proceedings of the Copenhagen Academy* for 1799, and is exceedingly clear and complete, even in comparison with modern works. He also considers the sphere, and gives a quaternion theory from which he develops a complete spherical trigonometry. In 1804 the Abbé Buée independently came upon the same idea which Wallis had suggested, that $\sqrt{-1}$ should represent a unit line, and $-\sqrt{-1}$ its negative, perpendicular to the real axis. Buée's paper was not published until 1806, in which year Jean-Robert Argand also published a pamphlet on the same subject. One is usually referred to Argand's essay for the scientific foundation for the graphic representation of complex numbers. Even with this background in 1831 Gauss found the theory quite unknown, and in 1832 published his chief memoir on the subject. His stature and this memoir brought this geometric representation prominently before the mathematical world.

The general acceptance of the theory is due in large part to the labors of Augustin Louis Cauchy and Niels Henrik Abel, especially the latter, who was the first to use complex numbers and achieve well-known success with the results.

[4]An *imaginary number* (or purely imaginary number) is a complex number whose square is a negative real number. Imaginary numbers were defined in 1572 by Rafael Bombelli. At the time, such numbers were thought not to exist, much as zero and the negative numbers were sometimes regarded by some as fictitious or useless. Many other mathematicians were slow to believe in imaginary numbers at first, including Descartes who wrote about them in his *La Gomtrie*.

The common terms used in the theory are chiefly due to the founders. Argand called $\cos\theta + i\sin\theta$ the *direction factor*, and $r = \sqrt{a^2+b^2}$ the *modulus*. Cauchy in his work in 1828 called $\cos\theta + i\sin\theta$ the *reduced form* (*l'expression réduite*). Gauss used i for $\sqrt{-1}$, introduced the term *complex number* for $a+bi$, and called a^2+b^2 the *norm*. The expression *direction coefficient*, sometimes used for $\cos\theta + i\sin\theta$, is due to Hankel (1867), and the *absolute value*, for *modulus*, is due to Weierstrass.

The field of complex numbers has a very important property for the study of algebra and, hence, it has implications for calculus. The property is that of *algebraic closure* and this is captured in the *Fundamental Theorem of Algebra*. This theorem answers a question that drove most of the field of mathematics in the Renaissance and later. It has to do with the dread of every high school student — factoring. Simply stated the question that one wanted answered was given any polynomial with integer coefficients, $p(x)$, can you solve the polynomial equation $p(x) = 0$. Since solutions or roots of a polynomial are intimately related to factors of that polynomial, then I can solve the equation if I can factor the polynomial.

We know that the Arabic mathematicians are credited with the development of early symbolic algebra and the solution to the quadratic equation is attributed to them. It is true that the Babylonians seemed to have had an early method for solving some quadratic equations, but the Islamic technique solved them all. It did occasionally give these "fictitious" roots, but mathematicians were willing to accept that in order to solve the equations.

In the Renaissance mathematicians in Italy found solutions to cubic equations. There are many stories about who, how, when, and what. For this time suffice it to say that by the 1600's there was a general solution to every cubic equation. Again, as noted above, they may give these "fictitious" roots. The solution to the cubic was published by Gerolamo Cardano (1501-1576) in his treatise *Ars Magna*. However, Cardano was not the original discoverer of the result. The hint for the cubic had been provided by Niccol Tartaglia. Tartaglia himself had probably caught wind of the solution from another source. The solution was apparently first arrived at by a little-remembered professor of mathematics at the University of Bologna by the name of Scipione del Ferro (ca. 1465-1526). While del Ferro did not publish his solution, he disclosed it to his student Antonio Maria Fior. This is apparently where Tartaglia learned of the solution around 1541.

A general quartic equation (also called a biquadratic equation) is a fourth-order polynomial equation of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + x^4 = 0.$$

Ferrari was the first to develop an algebraic technique for solving the general quartic, which was stolen and published in Cardano's *Ars Magna* in 1545.

So, by the end of the 16th century we are able to solve:
Linear equations — $ax + b = 0$ — $x = -\frac{b}{a}$

Quadratic equations — $x^2 + ax + b = 0$ — $y = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

Cubic equations — $x^3 + ax^2 + bx + c = 0$ —

$$\begin{aligned}
r_1 &= -\frac{a}{3} + \left(\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3} \\
&\quad + \left(\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3} \\
r_2 &= -\frac{a}{3} - \frac{1 + i\sqrt{3}}{2}\left(\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3} \\
&\quad + \frac{-1 + i\sqrt{3}}{2}\left(\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3} \\
r_3 &= -\frac{a}{3} + \frac{-1 + i\sqrt{3}}{2}\left(\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3} \\
&\quad - \frac{1 + i\sqrt{3}}{2}\left(\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(-a^2 + 3b)^3}}{54}\right)^{1/3}
\end{aligned}$$

Quartic (or biquadratic) equations — $x^4+ax^3+bx^2+cx+d = 0$ — See Appendix E.

The mathematical community was set to spend eternity solving each next higher order polynomial and finding more and more involved formulæ for the roots. In 1827 Niels Henrik Abel (Norway) proved that there was no general formula for solving the quintic equation in terms of radicals. This and the work of Evariste Galois in 1832 proved that there was no general formula for polynomials of degree n.

Thus, even though we cannot give a formula for factoring every polynomial, why would we think that we could? This is the strength of the Fundamental Theorem of Algebra.

Theorem 2.3 (Fundamental Theorem of Algebra) *Every polynomial of degree n with complex coefficients has n complex roots.*

A number of mathematicians had attempted to prove this starting with Euler in 1749. There were "issues" with the rigor of that proof as well as with those of Lagrange in 1772 and Laplace in 1795. Gauss in 1799 gave the first rigorous proof (along with a review of the other proofs and why he objected to them). His proof also had some problems, but was basically correct. In 1816 Gauss published a third proof of the theorem and this one was unequivocally correct.

One of the things that this does tell us is that every polynomial over the reals must factor into a product of linear terms and quadratic terms — a fact we use often in Algebra.

2.9.2 Hamiltonians and Quaternions

The quaternions are members of a noncommutative quaternions are a non-commutative extension of complex numbers first invented by William Rowan Hamilton . The idea for quaternions occurred to him while he was walking along the Royal Canal on his way to a meeting of the Irish Academy, and Hamilton was so pleased with his discovery that he scratched the fundamental formula of quaternion algebra,

$$i^2 = j^2 = k^2 = ijk = -1$$

into the stone of the Brougham bridge. The set of quaternions is denoted $\mathbb{H}$, or Q_8, and the quaternions are a single example of a more general class of hypercomplex numbers discovered by Hamilton. While the quaternions are not commutative, they are associative, and they form a group known as the quaternion group.

By analogy with the complex numbers being representable as a sum of real and imaginary parts, $a \cdot 1 + b \cdot i$, a quaternion can also be written as a linear combination

$$H = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k.$$

Quaternions are often used in computer graphics (and associated geometric analysis) to represent rotations and orientations of objects in three-dimensional space. They are smaller than other representations such as matrices, and operations on them such as composition can be computed more efficiently. Quaternions also see use in control theory, signal processing, attitude control, physics, and orbital mechanics, mainly for representing rotations/orientations in three dimensions. For example, it is common for spacecraft attitude-control systems to be commanded in terms of quaternions, which are also used to telemeter their current attitude. The rationale is that combining many quaternion transformations is more numerically stable than combining many matrix transformations, avoiding such phenomena as gimbal lock, which can occur when Euler angles are used. Using quaternions also reduces overhead from that when rotation matrices are used, because one carries only four components, not nine, the multiplication algorithms to combine successive rotations are faster, and it is far easier to renormalize the result afterwards.

2.9.3 Cayley numbers (Octonions)

The set of octonions, also called Cayley numbers and denoted $\mathbb{O}$, consists of the elements in a Cayley algebra. A typical octonion is of the form

$$a + bi_0 + ci_1 + di_2 + ei_3 + fi_4 + gi_5 + hi_6,$$

where each of the triples (i_0, i_1, i_3), (i_1, i_2, i_4), (i_2, i_3, i_5), (i_3, i_4, i_6), (i_4, i_5, i_0), (i_5, i_6, i_1), and (i_6, i_0, i_2) behaves like the quaternions (i, j, k). Octonions are not associative. They have been used in the study of eight- and eight-dimensional space.