Chapter 7

Series Examples

We have lots of theorems about convergence and divergence of series. How do we use them? Which one should be used when? Why all the fuss?

First, we have a result that tells us how closely related the Ratio Test and the Root Test actually are.

Theorem 7.1 *Let $\{a_n\}$ be any sequence of nonzero real numbers, then*

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

Corollary 7.1 *If $\lim |\frac{a_{n+1}}{a_n}|$ exists and equals L, then $\lim |a_n|^{1/n}$ exists and equals L.*

Example 7.1 Consider the series

$$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \cdots$$

This is a geometric series. It has the form that we desire, $\sum_{n=0}^{\infty} ar^n$ if we factor out $\frac{1}{9}$.

$$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{9}\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{9}\frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{12}.$$

This series could also be shown to converge by the Comparison Test, since $\sum 1/3^n$ converges by either the Ratio Test or the Root Test. In fact, if $a_n = (-1/3)^n$ then

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \limsup |a_n|^{1/n} = \frac{1}{3} < 1.$$

Of course none of these three tests gives us the actual sum as does the first procedure.

Example 7.2 Consider the series

$$\sum \frac{n}{n^2+3}.$$

If $a_n = \dfrac{n}{n^2+3}$, then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)^2+3} \cdot \frac{n^2+3}{n} = \frac{n+1}{n} \frac{n^3+3}{n^2+2n+4}$$

Hence, $\lim |a_{n+1}/a_n| = 1$. The Ratio Test will give us no information and you can check, but the Root Test will yield no information either. Before trying the Comparison Test, we should have some feeling as to whether it should converge or diverge. Since a_n is about $1/n$ for n large, we expect it to diverge. That gives us some indication of what to try to do.

$$\frac{n}{n^2+3} \geq \frac{n}{n^2+3n^3} = \frac{n}{4n^2} = \frac{1}{4n}.$$

Now, since $\sum 1/n$ diverges, $\sum 1/4n$ will diverge, so by the Comparison Test, our series diverges.

Example 7.3 Consider the series

$$\sum \frac{1}{n^2+1}$$

Neither the Ratio Test nor the Root Test gives any information. The Comparison Test works here because $\sum 1/n^2$ converges and $\dfrac{1}{n^2+1} \leq \dfrac{1}{n^2}$, so our series converges.

Example 7.4 Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

This time we still think that this converges because it looks like $\sum 1/n^2$. However, we cannot use the Comparison Test because

$$\frac{1}{n^2} \leq \frac{1}{n^2-1}$$

which is the wrong direction for using the Comparison Test. This is where the Limit Comparison Test comes into play. If $a_n = \dfrac{1}{n^2-1}$ and $b_n = \dfrac{1}{n^2}$ then

$$r = \lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since the limit exists, then $\sum 1/(n^2-1)$ converges because $\sum 1/n^2$ converges.

Example 7.5 Consider the series

$$\sum \frac{n}{3^n}$$

Using the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{3^{n+1}}\cdot\frac{3^n}{n} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{3^n}{3^{n+1}} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{3} = \frac{1}{3} < 1.$$

Thus, the series converges.

The Root Test gives

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{n}{3^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{1/n}}{3} = \frac{1}{3} < 1.$$

Example 7.6 Consider the series

$$\sum\left[\frac{2}{(-1)^n - 3}\right]^n$$

The form of a_n suggests that we use the Root Test. Note that if n is even, $|a_n|^{1/n} = 1$ while if n is odd, $|a_n|^{1/n} = \frac{1}{2}$. This means that $\limsup |a_n|^{1/n} = 1$, so the Root Test gives no information, nor will the Ratio Test. However we should look at each entry, if n is even, then $a_n = 1$. Thus, the terms never get arbitrarily close to 0, so the series diverges by Theorem **??**.

Example 7.7 Consider the series

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots$$

Let $a_n = 2^{(-1)^n - n}$. Since $a_n \le \frac{1}{2^{n-1}}$ for all n, the series converges by the Comparison Test. That is not the real interest in this series though. Let's look at what the Ratio Test tells us.

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{8} & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

Therefore,

$$\frac{1}{8} = \liminf\left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup\left|\frac{a_{n+1}}{a_n}\right| = 2,$$

and the Ratio Test gives us no information.

Now,

$$(a_n)^{1/n} = \begin{cases} 2^{\frac{1}{n}-1} & \text{if } n \text{ is even} \\ 2^{-\frac{1}{n}-1} & \text{if } n \text{ is odd} \end{cases}$$

Since $\lim 2^{\frac{1}{n}} = \lim 2^{-\frac{1}{n}} = 1$, we can conclude that $\lim a_n^{1/n} = \frac{1}{2} < 1$ and the series converges by the Root Test.

Example 7.8 Consider the series

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

Since $\lim \sqrt{n/(n+1)} = 1$, neither the Ratio Test nor the Root Test will give any information. Since $\sum \frac{1}{\sqrt{n}}$ diverges, we cannot use the Comparison Test to show our series converges. The Alternating Series Test shows that it converges, though.

Example 7.9 Consider the series

$$\sum \frac{n!}{n^n}$$

Should you use the Ratio Test or the Root Test?

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \frac{n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{1} \\
&= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \\
&= \frac{1}{\left(\frac{n+1}{n}\right)^n} \\
&= \frac{1}{\left(1+\frac{1}{n}\right)^n} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \\
&= \frac{1}{e}
\end{aligned}$$

Thus, the Ratio Test tells that the series converges. In fact, we can see that

$$\sum \frac{a^n n!}{n^n}$$

converges for $a < e$ and diverges for $a > e$. Note that from this and from our first theorem it must be true that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

or $\sqrt[n]{n!}$ is approximately equal to n/e for n large.