Chapter 9

Special Functions

We want to look at some special functions that can arise, especially in trying to solve certain types of rather simple equations.

9.1 Hyperbolic Trigonometric Functions

The usual trigonometric functions with which we are familiar are often called *circular functions* because their values can be determined by the geometry of a circle. While mathematicians were able to compute the area and circumference of a circle exactly, they were unable to do the same for an ellipse.

If $\frac{x^2}{a^2} + y^2 b^2 = 1$ describes an ellipse centered at the origin with major and minor axes of lengths $2a$ and $2b$, then we can show that the area of the ellipse is πab. This is actually not too hard using simple integration. However, the problem of finding the arclength of the ellipse, or the *rectification* of an ellipse, turned out to be much, much harder. The functions that were encountered were much harder. The integrals that arose in this study were known as *elliptic integrals*. These were found to be of several kinds. Nonetheless, they all arose basically in the search for finding the arclength of an ellipse. One reason that the ellipse was of such interest was that with Newton's theory of gravity, scientists and mathematicians began to understand that the earth wasn't a sphere but and ellipsoid. Also, with Kepler's Laws of Planetary Motion, the motion of the planets around the sun were not circular, but elliptical. Computing how long it takes for a planet to go around the sun, means that we need to know how far it travels, which is the arclength of an ellipse.

Thus, the study of functions defined by different conics was not too surprising. One point of importance, though, is that the ellipse can be parameterized, just like the circle, with the usual trigonometric functions:

$$x(t) = a\cos(t) \qquad y(t) = b\sin(t).$$

The study of a hyperbola would not lend itself to that technique, unless we allow complex numbers. Since early on complex numbers were not well received, mathematicians looked for a different way to approach this.

Vincenzo Riccati (1707-1775) introduced the hyperbolic functions. Johann Heinrich Lambert (1728-1777) further developed the theory of hyperbolic functions in *Histoire de l'acadmie Royale des sciences et des belles-lettres de Berlin*, vol. XXIV, p. 327 (1768).

Vincent Riccati, S.J. was born in Castel-Franco, Italy. He worked together with Girolamo Saladini in publishing his discovery, the hyperbolic functions. Riccati not only introduced these new functions, but also derived the integral formulas connected with them, and then, still using geometrical methods. He then went on to derive the integral formulas for the trigonometric functions. His book *Institutiones* is recognized as the first extensive treatise on integral calculus. The works of Euler and Lambert came later.

Vincent Riccati developed the hyperbolic functions and proved their consistency using only the geometry of the unit hyperbola $x^2 - y^2 = 1$ or $2xy = 1$. Riccati followed his father's interests in differential equations which arose naturally from geometrical problems. This led him to a study of the rectification of the conics in Cartesian coordinates and to an interest in the areas under the unit hyperbola. He developed the properties of the hyperbolic functions from purely geometrical considerations.

Riccati developed the hyperbolic functions in a manner similar to what follows. The algebra concerning sectors of a circle is relatively simple since the arc length is proportional to the area of the sector.

$$\ell = r\theta, \quad A = r^2\frac{\theta}{2}, \text{ so } \ell = \frac{A}{r}.$$

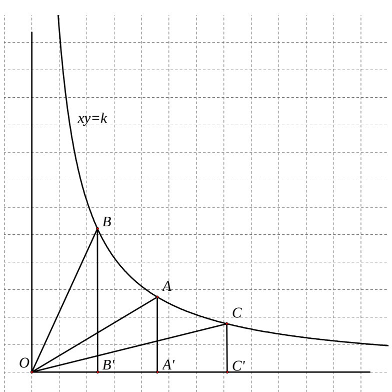

Figure 9.1:

Hyperbolas, however, do not have this property, so a different algebra is needed. If we look at the hyperbola $xy = k$, and let O denote the origin, A, a point on the graph and A' the foot of A on the x-axis, then the area of $\triangle OAA'$ is given by

$$A = \frac{1}{2}bh = \frac{1}{2}xy = \frac{k}{2}.$$

Thus, all such right triangles have the same area, no matter which point A we choose on the hyperbola. This property enables us to replace area measure by linear measure OA', thus introducing a kind of logarithm.

Let us now consider the curve $xy = 1$. It can be seen from Figure 9.2 that the following areas are equivalent: area($\triangle AOA'$) = area($\triangle POP'$) because of the property just mentioned, area($\triangle OPQ$) = area($\square QAA'P'$) by subtracting the common area $\triangle OQP'$ from each. Then area($APP'A'$) = area(AOP) by adding AQP to each. So for any point A on the curve, the area of the sector AOP equals the area under the curve $1/x$ from A to P. This area is $u = \ln a$ where $(0, a)$ corresponds to the point A'.

Now, we can use this parameter, u, to parameterize the original hyperbola, $x^2 - y^2 = 1$. For any point A on the hyperbola, let $u/2$ be the area of the region bounded by the x-axis, the hyperbola, and the line from the origin to A. The coordinates of the point A can be given by $(\cosh u, \sinh u)$. This is very similar to the situation for circular functions. If P is a point on the unit circle, then the area of the sector of the circle from the positive x-axis to the radius given by P is given by $\theta/2$ and the coordinates of the point P are $(\cos\theta, \sin\theta)$. This shows that the coordinates of the point on the unit circle can be determined by the area of the sector swept out by the radius.

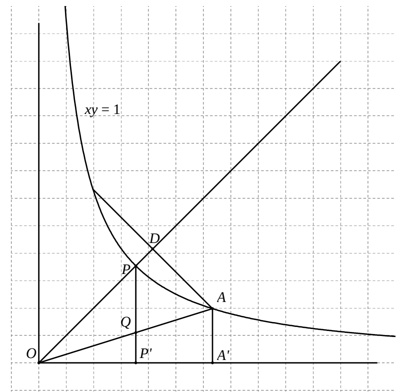

Figure 9.2:

The coordinates of the circle are then determined by the angle. This is not quite true for the hyperbola, because the hyperbola is asymptotic to the line $y = x$. This means that the angle determined by the x-axis, the origin, and a point on the hyperbola must always be less than $\pi/4$, or 45°. Likewise, as mentioned above the coordinates of a point on the circle can be parameterized by the arclength. For the hyperbola, the arclength is not u, but is bigger than that.

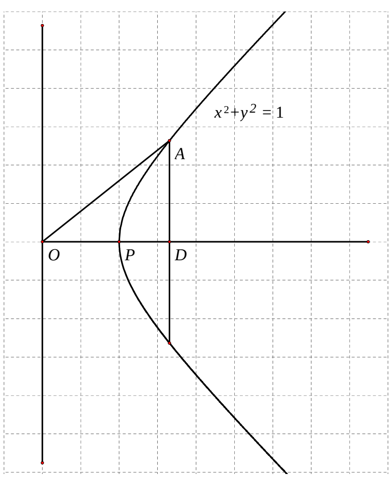

Figure 9.3:

We define the hyperbolic trigonometric functions by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$, the power series expansions of the hyperbolic trigonometric functions are

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and converge for all real x. In fact, using complex analysis and letting $i = \sqrt{-1}$, we can show that

$$\sinh x = -i\sin(ix) = i\sin\left(\frac{x}{i}\right)$$
$$\cosh x = \cos(ix) = \cos\left(\frac{x}{i}\right)$$

There are also the usual collections of hyperbolic trigonometry identities:

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

To understand the graphs of the hyperbolic sine and cosine functions, we first note that, for any value of x,

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\sinh(x),$$

and

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \cosh(x).$$

Now for large values of x, $e^{-x} \approx 0$, in which case $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \approx \frac{1}{2}e^x$ and $\sinh(-x) = -sinh(x) \approx -\frac{1}{2}e^x$. Thus the graph of $y = \sinh(x)$ appears as in Figure 9.4. Similarly, for large values of x, $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \approx \frac{1}{2}e^x$ and $\cosh(-x) = \cosh(x) \approx \frac{1}{2}e^x$. The graph of $y = \cosh(x)$ is shown in Figure 9.5.

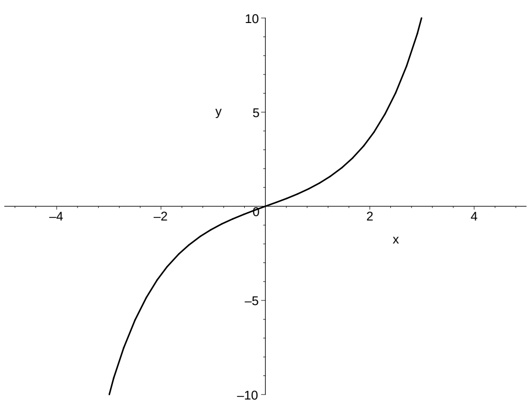

Figure 9.4: $y = \sinh x$

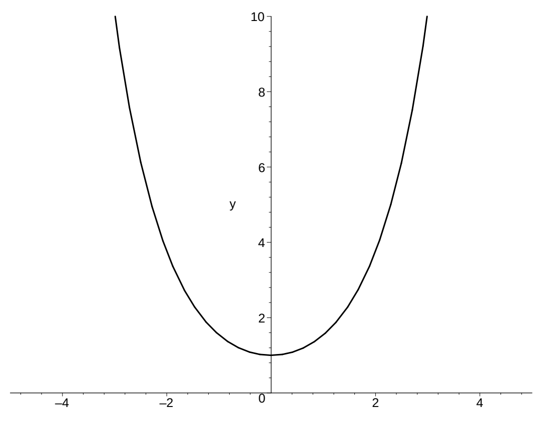

Figure 9.5: $y = \cosh x$

The derivatives of the hyperbolic sine and cosine functions follow immediately from their definitions. Namely,

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$
$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Since $\frac{d}{dx}\sinh(x) = \cosh(x) > 0$ for all x, the hyperbolic sine function is increasing on the interval $(-1, 1)$. Thus it has an inverse function, called the *inverse hyperbolic sine function*, with value at x denoted by $\sinh^{-1}(x)$. Since the domain and range of the hyperbolic sine function are both $(-\infty, \infty)$, the domain and range of the inverse hyperbolic sine function are also both $(-\infty, \infty)$. The graph of $y = \sinh^{-1}(x)$ is shown in Figure 9.6.

As usual with inverse functions, $y = \sinh^{-1}(x)$ if and only if $\sinh(y) = x$. This means that

$$\begin{aligned} x = \sinh(y) &= \frac{1}{2}(e^y - e^{-y}) \\ 2xe^y &= e^{2y} - 1 \\ e^{2y} - 2xe^y - 1 = 0 & \\ (e^y)^2 - 2xe^y - 1 = 0 & \\ e^y &= \frac{1}{2}(2x + \sqrt{4x^2 + 4} \\ y = \ln\left(x + \sqrt{x^2 + 1}\right) & \end{aligned}$$

where we chose the positive branch of the square root because e^y is never negative.

Note that

$$\sqrt{1 + x^2} = \sqrt{1 + \sinh^2(u)} = \sqrt{\cosh^2(u)} = \cosh(u),$$

This means that many integrals take the form of the following:

$$\int \frac{1}{\sqrt{1 + x^2}} = \int \frac{1}{\cosh u} \cosh u \, du = u + C = \sinh^{-1}(x) + C,$$

instead of

$$\int \frac{1}{\sqrt{1 + x^2}} = \ln\left(x + \sqrt{x^2 + 1}\right) + C_1.$$

Similarly, since $\frac{d}{dx}\cosh(x) = \sinh(x) > 0$ for all $x > 0$, the hyperbolic cosine function is increasing on the interval $[0, \infty)$, and so has an inverse if we restrict its domain to $[0, \infty)$. That is, we define the *inverse hyperbolic cosine function* by the relationship

$$y = \cosh^{-1}(x) \text{ if and only if } x = \cosh(y).$$

where we require $y \geq 0$. Note that since $\cosh(x) \geq 1$ for all x, the domain of of the inverse hyperbolic cosine function is $[1, \infty)$. The graph of $y = \cosh^{-1}(x)$ is shown in Figure 9.7.

It will come as no surprise then that we can show that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$. Likewise, it should come as no surprise that we can define the other four hyperbolic

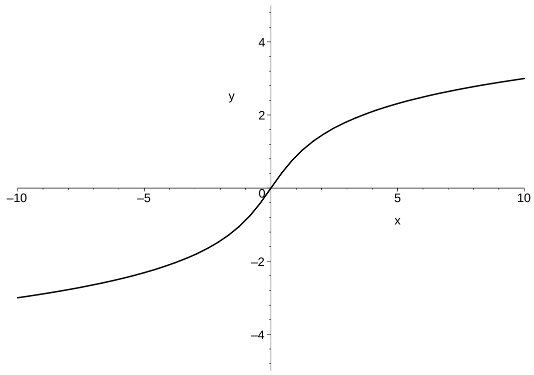

Figure 9.6: $y = \sinh^{-1} x$

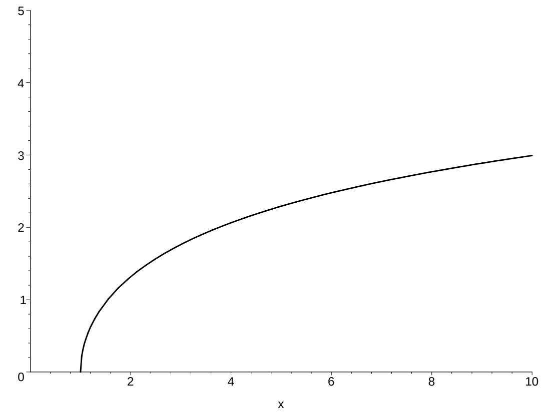

Figure 9.7: $y = \cosh^{-1} x$

functions as:

$$\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \\
\coth x &= \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \ x \neq 0 \\
\operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \\
\operatorname{csch} x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \ x \neq 0
\end{aligned}$$

These functions have the following properties:

$$\begin{aligned}
\frac{d}{dx} \sinh x &= \cosh x \\
\frac{d}{dx} \cosh x &= \sinh x \\
\frac{d}{dx} \tanh x &= \operatorname{sech}^2 x \\
\frac{d}{dx} \coth x &= -\operatorname{csch}^2 x \\
\frac{d}{dx} \operatorname{sech} x &= -\operatorname{sech} x \tanh x \\
\frac{d}{dx} \operatorname{csch} x &= -\operatorname{csch} x \coth x
\end{aligned}$$

Each one of these functions has an inverse function. We will look only at the inverse hyperbolic tangent function. As usual, define $y = \tanh^{-1} x$ to mean that $x = \tanh y$. The tangent function and its inverse are graphed in Figure 9.8. As with the other inverse hyperbolic functions, we find that

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \ -1 < x < 1.$$

We need to note that

$$\lim_{x \to \infty} \tanh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1.$$

Also, $\lim_{x \to -\infty} \tanh x = -1$.

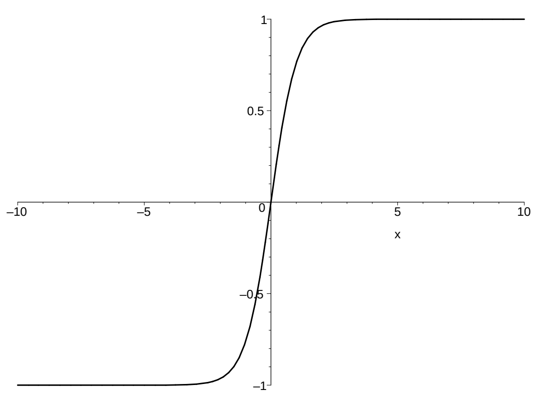

Figure 9.8: $y = \sinh^{-1} x$

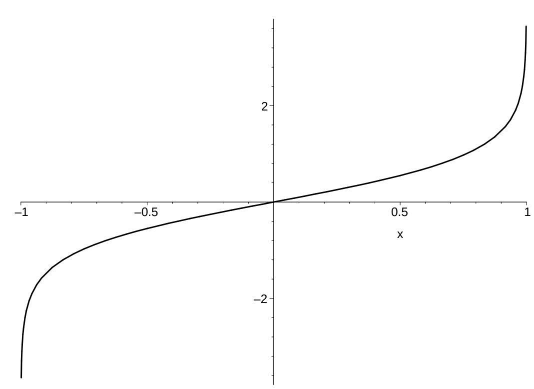

Figure 9.9: $y = \cosh^{-1} x$

The hyperbolic trigonometric functions mirror many of the same relationships as do the circular functions. A collection of the identities for the hyperbolic trigonometric functions follows:

$$\cosh^2 x - \sinh^2 x = 1 \tag{9.1}$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x \tag{9.2}$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{9.3}$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{9.4}$$
$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \tag{9.5}$$
$$\sinh^2 \tfrac{x}{2} = \frac{\cosh x - 1}{2} \tag{9.6}$$
$$\cosh^2 \tfrac{x}{2} = \frac{\cosh x + 1}{2} \tag{9.7}$$
$$\tanh^2 \tfrac{x}{2} = \frac{\cosh x - 1}{\cosh x + 1} \tag{9.8}$$
$$\tanh \tfrac{x}{2} = \frac{\sinh x}{\cosh x + 1} \tag{9.9}$$
$$= \frac{\cosh x - 1}{\sinh x} \tag{9.10}$$
$$\sinh x = 2 \sinh \tfrac{x}{2} \cosh \tfrac{x}{2} \tag{9.11}$$
$$\sinh x \pm \sinh y = 2 \sinh \tfrac{1}{2}(x \pm y) \cosh \tfrac{1}{2}(x \mp y) \tag{9.12}$$
$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y) \tag{9.13}$$
$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x + y) \sinh \tfrac{1}{2}(x - y) \tag{9.14}$$

9.2 Lambert W function

Johann Heinrich Lambert was born in Mulhouse on the 26th of August, 1728, and died in Berlin on the 25th of September, 1777. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. Lambert was the first to prove the irrationality of π. He worked on the parallel postulate, and also introduced the modern notation for the hyperbolic functions.

In a paper entitled *Observationes Variae in Mathesin Puram*, published in 1758 in *Acta Helvetica*, he gave a series solution of the trinomial equation, $x = q + x^m$, for x. His method was a precursor of the more general Lagrange inversion theorem. This solution intrigued his contemporary, Euler, and led to the discovery of the Lambert function.

Lambert wrote Euler a cordial letter on the 18th of October, 1771, expressing his hope that Euler would regain his sight after an operation; he explains in this letter how his trinomial method extends to series reversion.

The Lambert function is implicitly elementary. That is, it is implicitly defined by an equation containing only elementary functions. The Lambert function is not, itself, an elementary function. It is also not a *nice* function in the sense of Liouville, which means that it is not expressible as a finite sequence of exponentiations, root extractions, or antidifferentiations (quadratures) of any elementary function.

The Lambert function has been applied to solve problems in the analysis of algorithms, the spread of disease, quantum physics, ideal diodes and transistors, black holes, the kinetics of pigment regeneration in the human eye, dynamical systems containing delays, and in many other areas.

When we run into the equation $y = e^x$ we can solve this by using the inverse function $y = \ln x$. A similar equation that we find is $y = xe^x$. The graph of this function is in Figure 9.10.

Note that this function is not one-to-one, but is almost one-to-one. The minimal value is $-1/e$ which occurs at $x = -1$. If we restrict the domain to $[-1, \infty)$ or $(-\infty, -1]$ then there is an inverse on either branch. This function, the solution to the equation $y = xe^x$, is the LambertW function — named in honor of Lambert who first studied it, or one of its relative. Since the graph of the inverse is the reflection of the original graph across the line $y = x$. This looks like the graph in Figure 9.11.

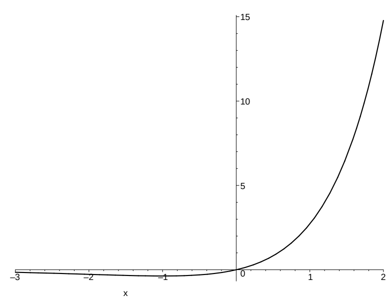

Figure 9.10: $y = xe^x$

This nomenclature was introduced by the developers of ***Maple*** when they found it to be very useful in solving a wide variety of problems.

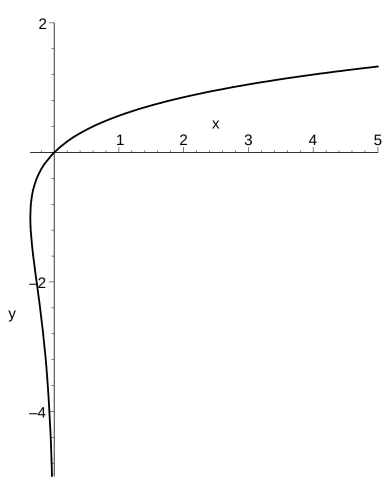

Figure 9.11: $y = W(x)$

Consider for example the equation $y = xe^{-x}$. The graph of this equation looks like the graph of $y = xe^x$ flipped across the y-axis and the x-axis. Thus, the solution would be $-W(-x)$. Note that $y = W(x)$ means that $x = ye^y$. Thus, $y = -W(-x)$ means

$$\begin{aligned} y &= -W(-x) \\ -y &= W(-x) \\ -x &= (-y)e^{-y} \\ x &= ye^{-y} \end{aligned}$$

The solution of $xe^x = a$ is $x = W(a)$ by definition, but Lémeray noted that a variety of other equations can be solved in terms of the same transcendental function. For example, the solution of $xb^x = a$ is $x = W(a \ln b)/\ln b$. The solution of $x^{x^a} = b$ is $e^{W(a \ln b)/a}$, and the solution of $a^x = x + b$ is $x = -b - W(-a^{-b} \ln a)/\ln a$.

Another example is rewriting the iterated exponentiation $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. Euler was the first to prove that this iteration converges for real numbers between e^{-e} and $e^{1/e}$. When it converges, it can be shown to converge to

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{W(-\ln x)}{-\ln x}.$$

It appears as if we do not know much about this function. However, we can learn quite a bit about it quickly. First, even though we can't write down the function explicitly, we can find it's derivative using implicit differentiation.

$$\begin{aligned} x &= W(x)e^{W(x)} \\ \frac{dx}{dx} &= \frac{d}{dx}\left(W(x)e^{W(x)}\right) \\ 1 &= W'(x)e^{W(x)} + W(x)e^{W(x)}W'(x) \\ W'(x) &= \frac{1}{e^{W(x)}(1 + W(x))} \end{aligned}$$

but $e^{W(x)} = x/W(x)$, so

$$W'(x) = \frac{W(x)}{x(1 + W(x))}, \ x \neq 0, -1/e$$

We can find a power series expansion for $W(x)$, but not in the usual way. What we know is a power series for xe^x. We would have

$$y = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} \ldots .$$

We want a power series for x in terms of y. The mathematician Lagrange was the first one to introduce the technique of *reversion of a series*. This is an algorithm that will allow for the computation of x as a series in y. It has been implemented in *Maple* and *Mathematica*. Instead of doing this, suffice it to say — for now — that the MacLaurin series for $W(x)$ is given by

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n.$$

(cf. Appendix H.)

Using integration by parts, we can show that

$$\int W(x)\,dx = \frac{x(W^2(x) - W(x) + 1)}{W(x)} + C.$$

This was discovered while working on a differential equation of the form

$$x = pe^p$$
$$\frac{dy}{dx} = p \text{ so } \frac{dx}{dy} = \frac{1}{p}$$

If we differentiate the first equation with respect to y, we get

$$\frac{dx}{dp} = \frac{dp}{dy}e^p + pe^p\frac{dp}{dy}$$
$$\frac{1}{p} = (p+1)e^p\frac{dp}{dy}$$
$$\frac{dy}{dp} = p(p+1)e^p$$

This can be easily integrated to get that

$$y = (p^2 - p + 1)e^p + C.$$

Since y is an antiderivative for $W(x)$ we get the above formula.

9.3 glog

Dr. Dan Kalman studied a generalized logarithm developed to solve *linear-exponential equations*, equations of the form $a^x = b(x+c)$ which do occur in Calculus and Numerical Analysis. His paper can be found in *The College Mathematics Journal*, Volume 32, Number 1, January 2001. His function, which he calls glog for generalized logarithm, is defined to be the solution to $xy = e^x$. This is very close to the Lambert W function and, in fact, he shows that his function:

$$\text{glog}(x) = -W(-1/x).$$

The graph reveals at once some of the gross features of glog. For one thing, glog is not a function because for $x > e$, it has two positive values. For $x < 0$ glog is well defined, and negative, but it is not defined for $0 \le x < e$.

When we need to distinguish between glog's two positive values, we will call the larger glog_+ and smaller glog_-.

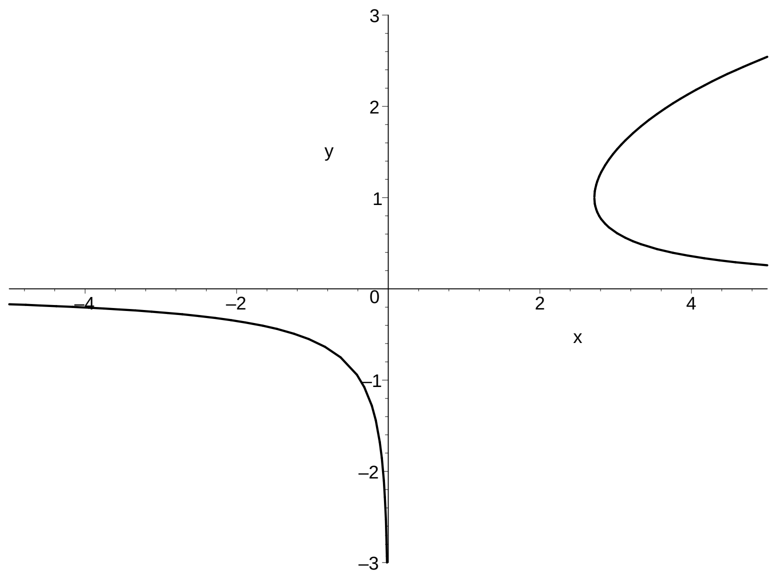

Figure 9.12: Graph of $y = \operatorname{glog} x$

These inequalities can be derived analytically. For example, suppose $y = \operatorname{glog} x$ is greater than 1. By definition, $e^y/y = x$ so $e^y = xy > x$. Thus $y > \ln x$.

The second graphical representation depends on the fact that $\operatorname{glog} c$ is defined as the solution to $e^x = cx$. The solutions to this equation can be visualized as the x-coordinates of the points where the curves $y = e^x$ and $y = cx$ intersect. That is, $\operatorname{glog} c$ is determined by the intersections of a line of slope c with the exponential curve.

When $c < 0$ the line has a negative slope and there is a unique intersection with the exponential curve. Lines with small positive slopes do not intersect the exponential curve at all, corresponding to values of c with no glog. There is a least positive slope at which the line and curve intersect, that intersection being a point of tangency. It is easy to see that this least slope is at $c = e$ with $x = 1$. For greater slopes there are two intersections, hence multiple solutions and the need for the two branches.