Chapter 11

Uniform Continuity

We saw in the exercises that there are some functions that are badly discontinuous, such as the characteristic function of the rationals on the reals:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

When we think of continuous functions, we tend to think of the usual functions from precalculus and calculus — polynomials, trigonometric functions, exponential functions, and so forth. These are continuous, yet somehow seem to be more than just meeting the definition of continuity.

By Theorem 10.1 we know that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on a set $S \subseteq \operatorname{dom}(f)$ if and only if

> for each $a \in S$ and $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in \operatorname{dom}(f)$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

From this definition we see that the choice of δ depends both on the point $a \in S$ and on the particular $\epsilon > 0$.

As an example, consider the function $f(x) = 1/x^2$ on the set $(0, +\infty)$. We know that f is continuous on this interval. Let $a > 0$ and $\epsilon > 0$. Now, we will need to show that $|f(x) - f(a)| < \epsilon$ for $|x - a|$ sufficiently small.

$$f(x) - f(a) = \frac{1}{x^2} - \frac{1}{a^2} = \frac{a^2 - x^2}{a^2x^2} = \frac{(a-x)(a+x)}{a^2x^2}.$$

If $|x - a| < \frac{a}{2}$, then $\frac{a}{2} < |x| < \frac{3a}{2}$ and $|x + a| < \frac{5a}{2}$. Thus, if $|x - a| < \frac{a}{2}$, then

$$|f(x) - f(a)| < \frac{|a - x| \cdot \frac{5a}{2}}{(\frac{a}{2})^2 x^2} = \frac{10|x - a|}{a^3}.$$

Thus if we let $\delta = \min\{\frac{a}{2}, \frac{a^3\epsilon}{10}\}$, then

$$|x - a| < \delta \text{ implies that } |f(x) - f(a)| < \epsilon.$$

Therefore, we have now shown that the conditions of Theorem 10.1 hold for f on $(0, +\infty)$. Note that δ depends on both ϵ and on a. Even if we fix ϵ, δ gets small when a is small. This shows that our choice of δ depends on the value of a as well as ϵ, though this might seem to be because of sloppy estimates. However, we can see that the value of δ must depend on a as well as ϵ.

It turns out that it is very useful to know when the δ in this condition can be chosen to depend only on $\epsilon > 0$ and the set S, so that δ does not depend on the particular point a.

Definition 11.1 *Let* $f: \mathbb{R} \to \mathbb{R}$ *be defined on* $S \subseteq \mathbb{R}$. *Then* f *is* uniformly continuous *on* S *if*

> *for each* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $x, y \in S$ *and* $|x - y| < \delta$ *then* $|f(x) - f(y)| < \epsilon$.

We will say that f *is* uniformly continuous *if it is uniformly continuous on* $\mathrm{dom}(f)$.

Note that this says that if f is uniformly continuous on S then for any given $\epsilon > 0$ the choice of $\delta > 0$ works for the entire set S.

Note that if a function is uniformly continuous on S, then it is continuous for every point in S. By its very definition it makes no sense to talk about a function being uniformly continuous at a point.

Now, we can show that the function $f(x) = 1/x^2$ is uniformly continuous on any set of the form $[a, +\infty)$. To do this we will have to find a δ that works for a given ϵ at every point in $[a, +\infty)$. We have

$$f(x) - f(y) = \frac{(y - x)(y + x)}{x^2 y^2}.$$

We want to see if we can prove that the term $\dfrac{x + y}{x^2 y^2}$ is bounded by some number M on $[a, +\infty)$. Once we have done that we can take $\delta = \epsilon / M$. Now,

$$\frac{x + y}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq \frac{1}{a^3} + \frac{1}{a^3} = \frac{2}{a^3}.$$

Thus, we will take

$$\delta = \frac{\epsilon a^3}{2}.$$

Question: How would we show that the function $g(x) = x^2$ is uniformly continuous on $[-5, 5]$?

Theorem 11.1 *If* f *is continuous on a closed interval* $[a, b]$, *then* f *is uniformly continuous on* $[a, b]$.

Proof: Assume that f is not uniformly continuous on $[a,b]$. Then there is an $\epsilon > 0$ such that for each $\delta > 0$ the implication

$$\text{“}|x-y| < \delta \text{ implies } |f(x)-f(y)| < \epsilon\text{”}$$

fails. Therefore, for each $\delta > 0$ there exists at least a pair of points $x, y \in [a,b]$ such that $|x-y| < \delta$ but $|f(x)-f(y)| \geq \epsilon$.

Thus, for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [a,b]$ so that $|x_n - y_n| < \frac{1}{n}$ but $|f(x) - f(y)| > \epsilon$. By the Bolzano-Weierstrauss Theorem (6.14) there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ that converges. Moreover, if $x_0 = \lim_{k\to\infty} x_{n_k}$, then $x_0 \in [a,b]$. Clearly we will also have to have that $x_0 = \lim_{k\to\infty} y_{n_k}$. Since f is continuous at x_0 we have

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}),$$

so

$$\lim_{k\to\infty} [f(x_{n_k}) - f(y_{n_k})] = 0.$$

Since $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for all k, we have a contradiction. This leads us to conclude that f is uniformly continuous on $[a,b]$. ∎

Note that in view of this theorem the following functions are uniformly continuous on the indicated sets: x^{45} on $[a,b]$, $\sqrt{x}$ on $[0,a]$, and $\cos(x)$ on $[a,b]$.

Theorem 11.2 *If f is uniformly continuous on A and $\{x_n\}$ is a Cauchy sequence in A, then $\{f(x_n)\}$ is a Cauchy sequence.*

Proof: Let $\{x_n\}$ be a Cauchy sequence in A and let $\epsilon > 0$. Since f is uniformly continuous on A, there is a $\delta > 0$ so that if $x, y \in A$ and $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$.

Since $\{x_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ so that if $m, n > N$ then $|x_m = x_n| < \delta$. Thus, this implies that if $m, n > N$ then $|f(x_m) - f(x_n)| < \epsilon$, which proves that $\{f(x_n)\}$ is a Cauchy sequence. ∎

As an example consider the function $f(x) = 1/x^2$ on $(0,1)$. Let $x_n = 1/n$ for $n \in \mathbb{N}$. This clearly forms a Cauchy sequence in $(0,1)$. However, the function takes the values $f(x_n) = n^2$ and the sequence $\{n^2\}$ is clearly not a Cauchy sequence. Thus, f cannot be a uniformly continuous function on $(0,1)$.

We define a function $\hat{f}$ to be an *extension* of f if $\text{dom}(f) \subseteq \text{dom}(\hat{f})$ and $f(x) = \hat{f}(x)$ for all $x \in \text{dom}(f)$.

Theorem 11.3 *A real-valued function f on (a,b) is uniformly continuous on (a,b) if and only if it can be extended to a continuous function $\hat{f}$ on $[a,b]$.*

PROOF: First, suppose that f can be extended to a continuous function $\hat{f}$ on $[a, b]$. Then $\hat{f}$ is uniformly continuous on $[a, b]$ by Theorem 11.1, so clearly f is uniformly continuous on (a, b).

Now, suppose that f is uniformly continuous on (a, b). We need to define $f(a)$ and $f(b)$ in such a way that the extension will be continuous. We will show how to deal with $\hat{f}(a)$ and the other extension is handled similarly.

Let $\{x_n\}$ be a sequence in (a, b) that converges to a. Since the sequence converges it must be a Cauchy sequence. Thus, $\{f(x_n)\}$ is also a Cauchy sequence. Therefore, it converges. Let's call this Condition A.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in (a, b) that both converge to a. Define a new sequence $\{u_n\}$ by interleaving x_n and y_n:

$$\{u_n\}_{n=1}^{\infty} = \{x_1, y_1, x_2, y_2, x_3, y_3, \ldots\}$$

It should be clear that $\lim_{n\to\infty} u_n = a$. Thus, $\lim_{n\to\infty} f(u_n)$ exists by Condition A. Since $\{f(x_n)\}$ and $\{f(y_n)\}$ are both subsequences of $\{f(u_n)\}$ they must converge and converge to the same limit. Thus,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n).$$

Let's call this Condition B.

Thus, define $\hat{f}(a) = \lim_{n\to\infty} f(s_n)$ for any sequence $\{x_n\}$ in (a, b) converging to a. Condition A guarantees that this limit exists, and Condition B guarantees that this limit is well-defined and unique. This implies that $\hat{f}$ is continuous at a. ∎

As an example consider the function $f(x) = \sin(x)/x$ for $x \neq 0$. We can extend this function on $\mathbb{R}$ by

$$\hat{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

The fact that $\hat{f}$ is continuous at $x = 0$ implies that f is uniformly continuous on $(a, 0)$ and $(0, b)$ for any $a < 0 < b$. In fact, $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

Theorem 11.4 *Let f be continuous on an interval I. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.*

PROOF: Let M be a bound for f' on I so that $|f'(x)| \leq M$ for all $x \in I^\circ$. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. Consider $a, b \in I$ where $a < b$ and $|b - a| < \delta$. By the Mean Value Theorem there exists $x \in (a, b)$ so that

$$f'(x) = \frac{f(b) - f(a)}{b - a},$$

so

$$|f(b) - f(a)| = |f'(x)| \cdot |b - a| \leq M|b - a| < M\delta = \epsilon.$$

Thus, f is uniformly continuous on I. ∎

Why is uniform continuity important? One of the reasons for studying uniform continuity is its application to the integrability of continuous functions on a closed interval, i.e. proving that a continuous function on a closed interval is integrable. To see how this might work with Riemann sums consider a continuous nonnegative real-values function f defined on $[0,1]$. For $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n-1$, let

$$M_{k,n} = \text{lub}\{f(x) \mid x \in [\frac{k}{n}, \frac{k+1}{n}]\}$$

$$m_{k,n} = \text{glb}\{f(x) \mid x \in [\frac{k}{n}, \frac{k+1}{n}]\}$$

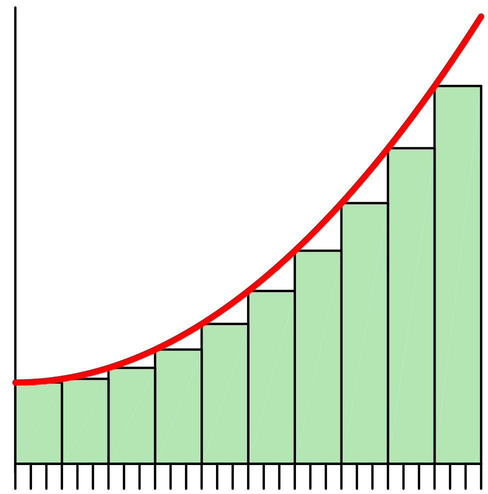

Figure 11.1: Lower Sums

Then the sum of the areas of the rectangles in Figure 11.2 equals

$$U_n = \frac{1}{n}\sum_{k=0}^{n-1} M_{k,n}$$

and the sum of the areas of the rectangles in Figure 11.1 equals

$$L_n = \frac{1}{n}\sum_{k=0}^{n-1} m_{k,n}.$$

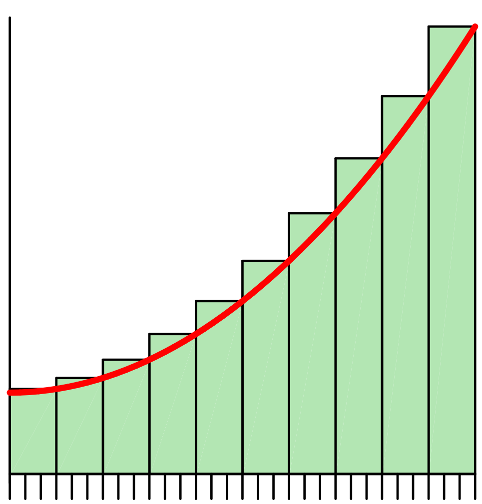

Figure 11.2: Upper Sums

The function f is Riemann integrable if the numbers U_n and L_n are close together for large n, in other words, if

$$\lim_{n\to\infty}(U_n - L_n) = 0.$$

In that case we define

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n.$$

In order to prove that the above limit is 0, we actually need uniform continuity. Note that

$$0 \le U_n - L_n = \frac{1}{n}\sum_{k=0}^{n-1}(M_{k,n} - m_{k,n})$$

for all n. Let $\epsilon > 0$. By our previous theorem, f is uniformly continuous on $[0,1]$, so there exists $\delta > 0$ so that

$$x, y \in [0,1] \text{ and } |x-y| < \delta \text{ imply } |f(x) - f(y)| < \epsilon.$$

Now, choose an N so that $\frac{1}{N} < \delta$. If $n > N$ then for $i = 0, 1, 2, \ldots, n-1$ we know that there exist $x_i, y_i \in [\frac{i}{n}, \frac{i+1}{n}]$ satisfying $f(x_i) = m_{i,n}$ and $f(y_i) = M_{i,n}$. Since $|x_i - y_i| \le \frac{1}{n} < \frac{1}{N} < \delta$, the above shows that $M_{i,n} - m_{i,n} = f(y_i) - f(x_i) < \epsilon$, so that

$$0 \le U_n - L_n = \frac{1}{n}\sum_{i=0}^{n-1}(M_{i,n} - m_{i,n}) < \frac{1}{n}\sum_{i=0}^{n-1}\epsilon = \epsilon.$$

Which proves the limit as desired.

11.1 Limits of functions

If f is continuous at $x = a$ we are tempted to write $\lim_{x\to a} f(x) = f(a)$ except that we have not defined how to find a limit of a function, only limits of sequences. We need to formalize the concept of a limit of a function at a point.

Since we will be interested in left-hand limits, right-hand limits, ordinary limits and limits at infinity, we will start with the following definition.

Definition 11.2 *Let $S \subseteq \mathbb{R}$, and let a be a real number or the symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or the symbol ∞ or $-\infty$. We write*

$$\lim_{x\to a^S} f(x) = L$$

if f is a function defined on S and fore every sequence $\{x_n\}$ in S with limit a we have $\lim_{n\to\infty} f(x_n) = L$.

This is a slightly different definition than that upon which we will eventually finalize. It has the advantage that we can continue to use the power of sequences, about which we know a lot.

Note that from our definition a function f is continuous at $a \in \mathrm{dom}(f) = S$ if and only if $\lim_{x\to a^S} f(x) = f(a)$. Also, note that the limits, when they exist, are unique. From this we will generate the usual definitions.

Definition 11.3

a) For $a \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ we write $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some set $S = J \setminus \{a\}$ where J is an open interval containing a. $\lim_{x\to a} f(x)$ is called the two-sided limit of f at a. *Note that f does not have to be defined at a and, even if f is defined at a, the value $f(a)$ does not have to be equal to the limit. In fact, $f(a) = \lim_{x\to a} f(x)$ if and only if f is defined on an open interval containing a and f is continuous at a.*

b) For $a \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ we write $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x\to a^+} f(x)$ is the right hand limit of f at a. *Again, f does not have to be defined at a.*

c) *For* $a \in \mathbb{R}$ *and* $f\colon \mathbb{R} \to \mathbb{R}$ *we write* $\lim_{x\to a^-} f(x) = L$ *provided* $\lim_{x\to a^S} f(x) = L$ *for some open interval* $S = (c, a)$. $\lim_{x\to a^-} f(x)$ *is the* left hand limit of f at a.

d) *For a function* $f\colon \mathbb{R} \to \mathbb{R}$ *we write* $\lim_{x\to\infty} f(x) = L$ *provided* $\lim_{x\to\infty^S} f(x) = L$ *for some open interval* $S = (c, \infty)$. *Likewise we write* $\lim_{x\to-\infty} f(x) = L$ *provided* $\lim_{x\to-\infty^S} f(x) = L$ *for some open interval* $S = (-\infty, b)$.

Theorem 11.5 *Let* f_1 *and* f_2 *be functions for which the limits* $\lim_{x\to a^S} f_1(x) = L_1$ *and* $\lim_{x\to a^S} f_2(x) = L_2$ *exist and are finite. THen*

i) $\lim_{x\to a^S}(f_1 + f_2)(x)$ *exists and equals* $L_1 + L_2$*;*

ii) $\lim_{x\to a^S}(f_1 f_2)(x)$ *exists and equals* $L_1 L_2$*;*

iii) $\lim_{x\to a^S}(f_1/f_2)(x)$ *exists and equals* L_1/L_2 *provides* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$.

Proof: The hypotheses imply that both f_1 and f_2 are defined on S and that a is the limit of some sequence in S. It is clear that the functions $f_1 + f_2$, $f_1 f_2$ and f_1/f_2 are defined on S, the latter if $f_2(x) \neq 0$ for $x \in S$.

Let $\{x_n\}$ be a sequence in S with limit a. By our hypotheses we have $L_1 = \lim_{n\to\infty} f_1(x_n)$ and $L_2 = \lim_{n\to\infty} f_2(x_n)$. By our theorems on convergent sequences we have that

$$\lim_{n\to\infty}(f_1 + f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) + \lim_{n\to\infty} f_2(x_n) = L_1 + L_2,$$

and

$$\lim_{n\to\infty}(f_1 f_2)(x_n) = \left[\lim_{n\to\infty} f_1(x_n)\right] \cdot \left[\lim_{n\to\infty} f_2(x_n)\right] = L_1 L_2.$$

Thus, condition (b) in the definition holds for $f_1 + f_2$ and $f_1 f_2$, so that (i) and (ii) hold. Part (iii) holds by a similar argument. ∎

Theorem 11.6 *Let* f *be a function for which the limit* $L = \lim_{x\to a^S} f(x)$ *exists and is finite. If* g *is a function define on the set* $\{f(x) \mid x \in S\} \cup \{L\}$ *that is continuous at* L*, then* $\lim_{x\to a^S} g \circ f(x)$ *exists and equals* $g(L)$.

Example 11.1 Why does g have to be continuous at $x = L$? Consider the following example. Let

$$f(x) = 1 + x\sin\frac{\pi}{x},\ x \neq 0 \quad \text{and} \quad g(x) = \begin{cases} 4 & x \neq 1 \\ -4 & x = 1 \end{cases}$$

Now, note that

$$\lim_{x\to 0} f(x) = 1 \quad \lim_{x\to 1} g(x) = 4$$

but what about $\lim_{x\to 0} g(f(x))$? Let $x_n = \frac{2}{n}$ for $n \in \mathbb{N}$, then

$$f(x_n) = 1 + \frac{2}{n}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 \pm \frac{2}{n} \neq 1 & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$g(f(x_n)) = \begin{cases} -4 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

Now, $\lim_{n\to\infty} x_n = 0$ so $\{x_n\}$ converges, but $\lim_{x\to 0} g(f(x))$ cannot exist.

Theorem 11.7 *Let f be a function defined on $S \subseteq \mathbb{R}$, let $a \in \mathbb{R}$ be a real number that is the limit of some sequence in S, and let L be a real number. Then $\lim_{x\to a^s} f(x) = L$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in S$ and $|x - a| < \delta$ then $|f(x) - L| < \epsilon$.*

Corollary 11.1 *Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a} f(x) = L$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.*

Corollary 11.2 *Let f be a function defined on some open interval (a, b), and let L be a real number. Then $\lim_{x\to a^+} f(x) = L$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \epsilon$.*

Theorem 11.8 *Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x)$ exists if and only if the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal, in which case all three limits are equal.*