Chapter 12

Sequences and Series of Functions-The Highlights

We won't have time this semester to study sequences and series of functions in any depth, but we will look a few of the highlights - important results - to see where this will head us in the future.

As I have said earlier, my framework for this course is making an analogue in the realm of functions to the construction of the real numbers from the natural numbers. First we start with the natural numbers (polynomials) from which we build the rational numbers (rational functions) and then we are led to real numbers (examples such as the transcendental functions). Along the way we saw that we were able to represent any real number as a decimal. In reality this decimal representation is a series of the form $\sum a_n/10^n$ where $a_n = 0, 1, \ldots, 9$. Is there such a thing as a "decimal representation" for the set of, say, continuous functions? What would it look like? How would we represent them?

We have mentioned in other places that we use polynomials to approximate other functions because they tend to be easier to evaluate. This is helpful in evaluating functions numerically using technology, because computers basically only add and multiply - unless you layer a filtering language over the machine code. This idea of approximating by polynomials has more ancient roots though. One of Newton's main contributions to calculus was his work on the binomial theorem:

$$(x+a)^n = \sum_{k=0}^{n} x^k a^{n-k}.$$

He begins, as did Wallis, by making area computations of the curves $(1-x^2)^n$, and tabulating the results. He noticed the Pascal triangle and reconstructed the formula

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

for positive integers n. Then to compute $\int_0^x \sqrt{1-x^2}\,dx$, he simply applied this relation with $n = 1/2$. This of course generated an infinite series because the terms do not terminate.

Next he generalized to function of the form $(a + bx)^n$ for any n including $n < 0$. This gave him the general binomial theorem - but not a proof.

He was able to determine the power series for $\ln(1 + x)$ by integrating the series for $(1 + x)^{-1}$, written as the binomial series. In modern notation, we have

$$\begin{aligned}(1+x)^{-1} &= 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \dots \\ &= 1 + \frac{-1}{1}x + \frac{-1\times -2}{2}x^2 + \left(\frac{-1\times -2\times -3}{3!}\right)x^3 + \dots \\ &= 1 - x + x^2 - x^3 \dots\end{aligned}$$

Now he integrated to get the series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \dots$$

With this he was able to compute logarithms of the number 1 ± 0.1, 1 ± 0.2, 1 ± 0.01 and 1 ± 0.02 to 50 places of accuracy. Then using identities such as

$$2 = \frac{1.2 \times 1.2}{0.8 \times 0.9}$$

he was able to compute the logarithm of many numbers.

In general the *binomial series* is the formal series of the form

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

in which

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(k-1))}{k!} = \frac{(-1)^k}{k!}(-\alpha)_k$$

with $\binom{\alpha}{0} = 1$ for any real number α. The series convergence depends on x and α.

- If $|x| < 1$, the series converges to $(1 + x)^\alpha$ for all $\alpha \in \mathbb{R}$.
- If $x = 1$, the series converges to 2^α for $\alpha > -1$.
- If $x = -1$, the series converges to 0 for $\alpha \geq 0$.

Note that for $\sqrt{1+x}$ this gives us:

$$\begin{aligned}
(1+x)^{1/2} &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k \\
&= \binom{\frac{1}{2}}{0} x^0 + \binom{\frac{1}{2}}{1} x + \binom{\frac{1}{2}}{2} x^2 + \binom{\frac{1}{2}}{3} x^3 + \ldots \\
&= 1 + \frac{\frac{1}{2}}{1!} x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} x^3 + \ldots \\
&= 1 + \frac{1}{2} x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} x^3 + \ldots \\
&= 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 + \ldots \\
(1+x)^{1/2} &= 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \frac{7}{256} x^5 - \frac{21}{1024} x^6 + \frac{33}{2048} x^7 - \ldots
\end{aligned}$$

The concept of power series has been around for quite awhile. Newton, however, did not consider the question of convergence. That concept was developed later.

12.1 Power Series

Given a sequence of real numbers, the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is called a *power series in x*. This power series is a "formal" definition. It does not consider the concept of convergence. However, if we are to consider functions, then we would say that the above power series is a function provided that it converges for some or all values of x. Note that it does converge for $x = 0$ and we will adopt the convention that $0^0 = 1$. Whether it converges for other values of x depends on the choice of coefficients a_n. Given any sequence $\{a_n\}$ one of the following holds for its power series:

a) the power series converges for all $x \in \mathbb{R}$;

b) the power series converges only at $x = 0$;

c) the power series converges for all x in some bounded interval centered at 0.

These follow from the following theorem.

Theorem 12.1 *For the power series $\sum a_n x^n$, let*

$$\beta = \limsup |a_n|^{1/n} \text{ and } R = \frac{1}{\beta}.$$

[If $\beta = 0$ we set $R = +\infty$ and if $\beta = +\infty$ we set $R = 0$.] Then

i) the power series converges for $|x| < R$*;*

ii) the power series converges for $|x| > R$*.*

R is called the *radius of convergence* for the power series.

Recall that if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then this limit equals β and the limit is often easier to calculate than the $\limsup |a_n|^{1/n}$.

Example 12.1 Consider

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

It is easy enough to check that

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1}{n+1} = 0.$$

Thus, $\beta = 0$, $R = +\infty$ and the series has an infinite radius of convergence. Thus, it converges for all $x \in \mathbb{R}$. Note,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

Example 12.2 Consider

$$\sum_{n=0}^{\infty} x^n.$$

It is easy enough to check that $\beta = 1$, $R = 1$ and the series has an radius of convergence of 1. We note that the series does not converge for $x = 1$ or $x = -1$. Thus the interval of convergence is $(-1, 1)$. Note,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Example 12.3 Consider

$$\sum_{n=0}^{\infty} \frac{1}{n} x^n.$$

Then $\beta = 1$, $R = 1$ and the series has an radius of convergence of 1. We note that the series does not converge for $x = 1$ since it is the harmonic series, but it does converge for $x = -1$ (Alternating Series Test). Thus the interval of convergence is $[-1, 1)$. Note,

$$\sum_{n=0}^{\infty} \frac{1}{n} x^n = \log(1-x).$$

Example 12.4 Consider

$$\sum_{n=0}^{\infty} \frac{1}{n^2} x^n.$$

Then $\beta = 1$, $R = 1$ and the series has an radius of convergence of 1. This series converges at both $x = 1$ and at $x = -1$. Thus the interval of convergence is $[-1, 1]$.

Example 12.5 Consider

$$\sum_{n=0}^{\infty} n! x^n.$$

Then $\beta = +\infty$ and $R = 0$. This series diverges for $x \neq 0$.

Example 12.6 Consider

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}.$$

This is deceptive since at first you will want to try to compute $\beta = \limsup (2^{-n})^{1/n} = \frac{1}{2}$ and $R = 2$. But this is incorrect! 2^{-n} is the coefficient of x^{3n} and we have to work with the coefficients of x^n. We have to be more careful here. We can write this series as $\sum_{n=0}^{\infty} a_n x^n$ where $a_{3k} = 2^{-k}$ and $a_n = 0$ if n is not a multiple of 3. Then you can calculate β by using the subsequence of all nonzero terms. This gives us

$$\beta = \limsup |a_n|^{1/n} = \lim_{k \to \infty} |a_{3k}|^{1/3k} = \lim_{k \to \infty} |2^{-k}|^{1/3k} = 2^{-1/3}.$$

Thus, the radius of convergence is $2^{1/3}$.

One of the major goals is to understand the function given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < R.$$

We want to know answers to such questions as: Is f continuous? Is f differentiable? If so, can you differentiate f term-by-term:

$$f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}?$$

Can you integrate a power series term-by-term?

These seem reasonable and for the most part seem like they might be true. That right there should make you pause. If we have learned anything, we know that we can usually create counterexamples to simple statements without sufficient conditions. That means we need to be careful.

Let's look at the question of continuity. The partial sums $f_n(x) = \sum_{k=0}^{n} a_k x^k$ are all continuous since they are polynomials and we have that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for } |x| < R.$$

Therefore, we would be in good shape if the following were true: If $\{f_n\}$ is a sequence of continuous functions on (a, b) and if $\lim f_n(x) = f(x)$ for all $x \in (a, b)$, then f is continuous on (a, b). Unfortunately, this is **false**.

Consider the following simple example. Let $f_n(x) = (1 - |x|)^n$ for $x \in (-1, 1)$. Let f be given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then $\lim f_n(x) = f(x)$ for all $x \in (-1, 1)$. Each f_n is continuous, but $f(x)$ is not continuous at $x = 0$.

12.2 Uniform Convergence

Definition 12.1 *Let $\{f_n\}$ be a sequence of real-valued functions defined on a set $U \subseteq \mathbb{R}$. The sequence $\{f_n\}$* converges pointwise *to $f(x)$ defined on U if*

$$\lim_{n\to\infty} f_n(x) = f(x) \textit{ for all } x \in U.$$

We will write $f_n \to f$ pointwise on U.

All of the above examples were pointwise convergence. Also $f_n(x) = x^n$ converges pointwise on [0,1] to the function $f(x)$ which is 0 on $[0, 1)$ and $f(1) = 1$.

Note that pointwise convergence says that for each $\epsilon > 0$ and $x \in U$ there is an $N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \epsilon$ whenever $n > N$. This value of N depends on ϵ and on $x \in U$. We want something stronger.

Definition 12.2 *Let $\{f_n\}$ be a sequence of real-valued functions defined on a set $U \subseteq \mathbb{R}$. The sequence $\{f_n\}$* converges uniformly *to $f(x)$ defined on U if for each $\epsilon > 0$ there is a number $N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \epsilon$ for all $x \in U$ and all $n > N$. In this case we write $f_n \to f$ uniformly on U.*

Note that if $f_n \to f$ uniformly on U and if $\epsilon > 0$ then there is an $N \in \mathbb{N}$ so that $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ for all $x \in U$ and $n > N$. In other words, the graph of f_n lies in a strip between $f(x) - \epsilon$ and $f(x) + \epsilon$ for all $x \in U$.

Theorem 12.2 *A uniform limit of continuous functions is continuous.*

Uniform convergence can be reformulated as follows. *A sequence $\{f_n\}$ of functions on a set $U \subseteq \mathbb{R}$ converges uniformly to a function f on U if and only if*

$$\lim_{n\to\infty}[\text{lub}\{|f(x) - f_n(x)| \mid x \in U\}] = 0.$$

This says that we can decide if a sequence converges uniformly by calculating this difference for each n. If $f - f_n$ is differentiable, we could use calculus to find these bounds.

Example 12.7 Consider

$$f_n(x) = \frac{x}{1+nx^2} \text{ for } x \in \mathbb{R}.$$

Now, $\lim f_n(0) = 0$. If $x \neq 0$ then $\lim(1+nx^2) = +\infty$ so that $f_n \to 0$ pointwise on $\mathbb{R}$. To find the maximum and minimum of f_n, find the critical points for f_n and classify them according to type. The critical points are $x = \frac{\pm 1}{\sqrt{n}}$. f_n takes is maximum at $\frac{1}{\sqrt{n}}$ and its minimum at $-\frac{1}{\sqrt{n}}$. Since $f(\pm\frac{1}{\sqrt{n}}) = \pm\frac{1}{2\sqrt{n}}$ we can conclude that

$$\lim_{n\to\infty}[\text{lub}\{|f_n(x)| \mid x \in U\}] = \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0.$$

Thus, $f_n \to 0$ uniformly on $\mathbb{R}$.

One strength of uniform continuity is the following result.

Theorem 12.3 *Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$, and suppose that $f_n \to f$ uniformly on $[a,b]$. Then*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

A sequence of functions $\{f_n\}$ defined on $U \subseteq \mathbb{R}$ is *uniformly Cauchy on U* if for each $\epsilon > 0$ there is a number $N \in \mathbb{N}$ so that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in U$ and all $m, n > N$.

This leads us to the following result.

Theorem 12.4 *Let $\{f_n\}$ be a sequence of functions defined and uniformly Cauchy on a set $U \subseteq \mathbb{R}$. Then there exists a function f on U so that $f_n \to f$ uniformly on U.*

This result is very useful in looking at series of functions.

Theorem 12.5 *Consider a series $\sum g_k$ of functions on a set $U \subseteq \mathbb{R}$. Suppose that each g_k is continuous on S and that the series converges uniformly on U. Then the series represents a continuous function on U.*

This leads to the ***Weierstrauss M-Test***.

Theorem 12.6 (Weierstrauss M-Test) *Let $\{M_k\}$ be a sequence of non-negative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all $x \in U$, then $\sum g_k$ converges uniformly on U.*

Lemma 12.1 *If the series $\sum g_k$ converges uniformly on a set $U \subseteq \mathbb{R}$ then*

$$\lim_{n\to\infty} [\text{lub}\{|g_n(x)| \mid x \in U\}] = 0.$$

Example 12.8 Let $f(x) = \sum_{n=1}^{\infty} 2^{-}nx^n$. We can easily show that the radius of convergence is 2. It is also clear that the series does not converge at $x = 2$ or at $x = -2$. Thus, the interval of convergence is $(-2, 2)$.

Let $0 < a < 2$. Note that $\sum_{n=1}^{\infty} 2^{-n}a^n = \sum_{n=1}^{\infty} (\frac{a}{2})^n$ converges. Since $|2^{-n}x^n| < 2^{-n}a^n = (\frac{a}{2})^n$ for $x \in [-a, a]$, the Weierstrauss M-Test shows that the series $\sum_{n=1}^{\infty} 2^{-n}x^n$ converges uniformly to a function on $[-a, a]$. By our previous theorem, the limit function f is continuous at each point of the set $[-a, a]$. Since a can be any number less than 2 we can conclude that f is continuous on $(-2, 2)$.

Now, $\text{lub}\{|2^{-n}x^n| \mid x \in (-2, 2)\} = 1$ for all n. Thus, by the above corollary, the convergence of the series cannot be uniform on $(-2, 2)$.

All of this tells us the following:

Theorem 12.7 *Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

As a corollary of this we see that the power series converges to a continuous function on the open interval $(-R, R)$. However, from our previous example we see that the series need not converge uniformly on its interval of convergence.

Lemma 12.2 *If the power series $\sum a_n x^n$ has a radius of convergence R, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

The next result indicates that a power series can be integrated term-by-term inside it interval of convergence.

Theorem 12.8 *Suppose $f(x) = \sum a_n x^n$ has radius of convergence $R > 0$. Then*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ for } |x| < R.$$

The next theorem indicates that we can differentiate term-by-term also.

Theorem 12.9 *Suppose $f(x) = \sum a_n x^n$ has radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ for } |x| < R.$$

Suppose that a power series has a radius of convergence greater than 1, and let f denote the function given by this power series. The first theorem of this section tells us that the partial sums of the power series get uniformly close to f on $[-1, 1]$. In other words, f can be approximated uniformly on $[-1, 1]$ by polynomials. What if you are given a function first? Can any function be approximated by polynomials?

Theorem 12.10 (Weierstrauss Approximation Theorem) *Every continuous function on a closed interval $[a, b]$ can be uniformly approximated by polynomials on $[a, b]$.*

S.N. Bernstein actually gave a constructive proof of this on $[0, 1]$. For a function f on $[0, 1]$, define the Bernstein polynomials $B_n f$ by

$$B_n f(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}.$$

He then proved that $B_n f \to f$ uniformly on $[0, 1]$.