Questions for Final Exam
MATH 6101/8101 - Fall 2006

20-November-2006

1. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all natural numbers $n \in \mathbb{N}$.

2. Prove that $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1) = 3n^2$ for all natural numbers $n \in \mathbb{N}$.

3. For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

   (a) Prove that P_{n+1} is true whenever P_n is true.

   (b) For which n is P_n actually true? What is the moral of this exercise?

4. If $A \subseteq \mathbb{R}$ we define the *maximum element of* A as the largest element in A; i.e.,

$$\max\{A\} = x \text{ means that } x \in A \text{ and if } a \in A \text{ then } a \leq x.$$

   Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that if $\operatorname{lub} S$ belongs to S, then $\operatorname{lub} S = \max\{S\}$.

5. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

   (a) Prove that if $S \subseteq T$, then $\operatorname{glb} T \leq \operatorname{glb} S \leq \operatorname{lub} S \leq \operatorname{lub} T$.

   (b) Prove that $\operatorname{lub}(S \cup T) = \max\{\operatorname{lub} S, \operatorname{lub} T\}$. Note, ***do not*** assume that $S \subseteq T$.

6. (a) Give an example of a bounded sequence or real numbers that does not converge.

   (b) Give an example of a bounded sequence of rational numbers that converges to an irrational number.

   (c) Give an example of a bounded sequence of irrational numbers that converges to a rational number.

   (d) Give an example of a bounded sequence of irrational numbers that converges to an irrational number.

7. Let $x_1 = 1$ and $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$ for $n \geq 1$. Assume that $\{x_n\}$ converges and find the limit.

8. (a) Prove that if A and B are countable sets then $A \times B$ is countable.

   (b) Prove that if A, B, and C are countable sets, then $A \times B \times C$ is countable.

   (c) Prove that if A_i is a countable set for $i = 1, 2, \ldots, n$ then $\prod_{i=1}^{n} A_i$ is countable.

   (d) Does you proof show that $\prod_{i=1}^{\infty} A_i$ is countable? Is it true?

9. Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

10. Let $\{a_n\}$ be a sequence of nonzero real numbers such that the sequence $\{\frac{a_{n+1}}{a_n}\}$ is a constant sequence. Show that $\sum a_n$ is a geometric series.

11. (a) Prove that the function $f(x) = kx$, $k \in \mathbb{R}$, is a continuous function on $\mathbb{R}$.
    (b) Prove that the function $g(x) = |x|$ is a continuous function on $\mathbb{R}$.

12. Prove that the function

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not continuous at $x_0 = 0$.

13. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there is an x between a and b such that $f(x) = 0$.

14. Suppose that f is continuous on $[0, 2]$ and the $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ so that $|y - x| = 1$ and $f(x) = f(y)$.
    Hint: Consider the function $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.