MATH 6101-090

ASSIGNMENT 1 - SOLUTIONS

September 11, 2006

1. *Using the Trichotomy Law prove that if a and b are real numbers then one and only one of the following is possible: $a < b$, $a = b$, or $a > b$.*

   Since a and b are real numbers then $a - b$ is a real number. By the Trichotomy Law we know that $a - b < 0$, $a - b = 0$ or $a - b > 0$. These immediately translate into $a < b$, $a = b$ or $a > b$.

2. *We define the* absolute value *of a real number a by*

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

   Prove the following:

   (a) $|a + b| \leq |a| + |b|$.

   We will each of these by cases. The case where either $ab = 0$ is not interesting, so we will leave it. We must have $a < 0$ or $a > 0$ and $b < 0$ or $b > 0$. Thus, we are left with 4 cases to check: (1) $a > 0$ and $b > 0$, (2) $a < 0$ and $b > 0$, (3) $a > 0$ and $b < 0$, and (4) $a < 0$ and $b < 0$.

   In case (1) since both a and b are positive, $a + b$ is positive and $|a| = a$, $|b| = b$, and $|a+b| = a+b$. Therefore $|a+b| = a+b = |a|+|b|$ and the statement is true.

   In case (2), since $a < 0$, $|a| = -a$. To show that $|a + b| \leq |a| + |b|$ we must show that

$$|a| + |b| - |a + b| \geq 0.$$

   Either $a + b \leq 0$ or $a + b \geq 0$.

   If $a + b \geq 0$

$$\begin{aligned} |a| + |b| - |a + b| &= (-a) + b - (a + b) \\ &= -2a > 0 \text{ since } -a > 0 \end{aligned}$$

   If $a + b \leq 0$

$$\begin{aligned} |a| + |b| - |a + b| &= (-a) + b - (-(a + b)) \\ &= (-a) + b + a + b)) \\ &= 2b > 0 \text{ since } b > 0 \end{aligned}$$

   Thus $|a + b| \leq |a| + |b|$ in this case.

   Case (3) is similar since the roles of a and b are reversed.

   Case (4) is similar to Case (1).

(b) $|xy| = |x| \cdot |y|$.

Here we break the proof up into the same cases: (1) $x > 0, y > 0$, (2) $x < 0, y > 0$, (3) $x > 0, y < 0$, and (4) $x < 0, y < 0$.

In Case (1) since $x > 0$ and $y > 0$, then $xy > 0$, and it easily follows that $|xy| = xy = |x| \cdot |y|$.

In Case (2) since $x < 0$ and $y > 0$, then $xy < 0$, and it easily follows that $|xy| = -(xy) = (-x)y = |x| \cdot |y|$.

In Case (3) since $x > 0$ and $y < 0$, then $xy < 0$, and it easily follows that $|xy| = -(xy) = x(-y) = |x| \cdot |y|$.

In Case (4) since $x < 0$ and $y < 0$, then $xy > 0$, and it follows that $|xy| = xy = (-x)(-y) = |x| \cdot |y|$.

(c) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$.

Since $x \neq 0$, we know that $\frac{1}{x}$ is its multiplicative inverse, so

$$1 = \left|x \cdot \frac{1}{x}\right| = |x| \cdot \left|\frac{1}{x}\right|.$$

Solving gives us that $\left|\frac{1}{x}\right| = \frac{1}{|x|}$.

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, if $y \neq 0$.

Use the above again and the fact that $\frac{x}{y} = x \cdot \frac{1}{y}$:

$$\left|\frac{x}{y}\right| = \left|x \cdot \frac{1}{y}\right| = |x| \cdot \left|\frac{1}{y}\right| = |x| \cdot \frac{1}{|y|} = \frac{|x|}{|y|}.$$

(e) $|x - y| \leq |x| + |y|$.

Solution Method I: The eloquent solution uses the results of Part 2a to show this:

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|,$$

where the inequality comes from 2a.

Solution Method II: You can do this one much like the first one. Break it into cases and do them one at a time. Cases: (1) $x > 0, y > 0$, (2) $x < 0, y > 0$, (3) $x > 0, y < 0$, and (4) $x < 0, y < 0$.

We need to show in each case that $|x| + |y| - |x - y| \geq 0$.

In Case (1) we have to deal with two cases $x - y \leq 0$ and $x - y \geq 0$. If $x - y \geq 0$, then $|x - y| = x - y$ and $|x| + |y| - |x - y| = x + y - (x - y) = 2y > 0$. If $x - y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x|+|y|-|x-y| = x+y-(y-x) = 2x > 0$. Thus, this is true in Case (1).

Case (2): In this case $|x| = -x$ and $|y| = y$. Again, we have to consider two cases: $x-y \leq 0$ and $x-y \geq 0$. However, note that if $x < 0$ and $y > 0$, it cannot happen that $x-y \geq 0$. So, $x-y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x|+|y|-|x-y| = -x+y-(y-x) = 0 \geq 0$. Thus, this is true in Case (2).

Case (3): In this case $|x| = x$ and $|y| = -y$. Again, we have to consider two cases: $x-y \leq 0$ and $x-y \geq 0$. Again, as in Case (2) it is impossible for $x-y \leq 0$. So, $x-y \geq 0$, then $|x-y| = x-y$ and $|x|+|y|-|x-y| = x-y-(x-y) = 0 \geq 0$. Thus, this is true in Case (3).

For Case (4), $|x| = -x$ and $|y| = -y$. If $x-y \geq 0$, then $|x-y| = x-y$ and $|x|+|y|-|x-y| = -x+(-y)-(x-y) = -2x > 0$. If $x-y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x|+|y|-|x-y| = -x+(-y)-(y-x) = -2y \geq 0$.

Thus, this is true.

(f) $|x|-|y| \leq |x-y|$.

Solution Method I: There is an eloquent solution here as well.

$$\begin{aligned} |x| &= |x-y+y| \\ &\leq |x-y|+|y| \\ |x|-|y| &\leq |x-y| \end{aligned}$$

Solution Method II: You can also break it into cases and do them one at a time. Cases: (1) $x > 0, y > 0$, (2) $x < 0, y > 0$, (3) $x > 0, y < 0$, and (4) $x < 0, y < 0$.

We need to show in each case that $|x-y|-(|x|-|y|) = |x-y|-|x|+|y| \geq 0$.

In Case (1) we have to deal with two cases $x-y \leq 0$ and $x-y \geq 0$. If $x-y \geq 0$, then $|x-y| = x-y$ and $|x-y|-|x|+|y| = x-y-x+y = 2x > 0$. If $x-y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x-y|-|x|+|y| = y-x-x+y = 2(y-x) > 0$. Thus, this is true in Case (1).

Case (2): In this case $|x| = -x$ and $|y| = y$. This time it is possible for $x-y \leq 0$ but impossible for $x-y \geq 0$. If $x-y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x-y|-|x|+|y| = y-x+x+y = 2y > 0$. Thus, this is true in Case (2),

Case (3): In this case $|x| = x$ and $|y| = -y$. This time it is possible for $x-y \geq 0$ but impossible for $x-y \leq 0$. If $x-y \geq 0$, then $|x-y| = x-y$ and $|x-y|-|x|+|y| = x-y-x-y = -2y \geq 0$. Thus, this is true in Case (3).

For Case (4), $|x| = -x$ and $|y| = -y$. If $x-y \geq 0$, then $|x-y| = x-y$ and $|x-y|-|x|+|y| = x-y-(-x)+(-y) = 2(x-y) > 0$. If $x-y \leq 0$, then $|x-y| = -(x-y) = y-x$ and $|x-y|-|x|+|y| = y-x-(-x)+(-y) = 0 \geq 0$.

Thus, this is true.

3. *The fact that $a^2 \geq 0$ for all real numbers a has tremendous implications. The most widely used of all inequalities is the* Schwarz inequality*:*

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

Do ONE of the following:

(a) *Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with*

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \qquad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$$

first for $i = 1$ and then for $i = 2$.

The first inequality comes from the fact that $0 \leq (x - y)^2 = x^2 - 2xy + y^2$, so $2xy \leq x^2 + y^2$. Thus, doing the algebra

$$\begin{aligned}
2xy &\leq x^2 + y^2 \\
2\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{y_1}{\sqrt{y_1^2 + y_2^2}} &\leq \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right)^2 + \left(\frac{y_1}{\sqrt{y_1^2 + y_2^2}}\right)^2 \\
2\frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} &\leq \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right)^2 + \left(\frac{y_1}{\sqrt{y_1^2 + y_2^2}}\right)^2 \\
2\frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} &\leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2} \\
&\text{and} \\
2\frac{x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} &\leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2} \\
\text{Adding these} & \\
2\frac{x_1y_1 + x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} &\leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} = 2 \\
\frac{x_1y_1 + x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} &\leq 1 \\
x_1y_1 + x_2y_2 &\leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}
\end{aligned}$$

(b) *Prove the Schwarz inequality by first proving that*

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

First,

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2.$$

Now,

$$\begin{aligned}(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2 &= x_1^2y_1^2+2x_1y_1x_2y_2+x_2^2y_2^2+x_1^2y_2^2-2x_1y_1x_2y_2+x_2^2y_1^2\\ &= x_1^2y_1^2+x_2^2y_1^2+x_1^2y_2^2+x_2^2y_2^2\\ &= (x_1^2+x_2^2)(y_1^2+y_2^2)\end{aligned}$$

$$\begin{aligned}(x_1^2+x_2^2)(y_1^2+y_2^2) &= (x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2\\ &\geq (x_1y_1+x_2y_2)^2\\ \text{Thus,}&\\ \sqrt{(x_1^2+x_2^2)}\sqrt{(y_1^2+y_2^2)} &\geq x_1y_1+x_2y_2\end{aligned}$$

and we are done.

4. Prove the following formulæ by induction

(a) $1^2+2^2+\cdots+n^2=\displaystyle\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$

First, check it for $n=1$ and we have $1^2=\frac{1\cdot2\cdot3}{6}=1$, so it is true for $n=1$. Now, assume it is true for k. We must prove that it is true for $k+1$.

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2+(k+1)^2\\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\ &= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}\\ &= \frac{(k+1)[2k^2+7k+6)]}{6}=\frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

which is what we needed, and we are done.

(b) $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$

First, check it for $n=1$ and we have $1^3=(1)^2$, so it is true for $n=1$. Now, assume it is true for k. We must prove that it is true for $k+1$.

$$\begin{aligned}
1^3+2^3+\cdots+(k+1)^3 &= 1^3+2^3+\cdots+k^3+(k+1)^3 \\
&= (1+2+\cdots+k)^2+(k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 = (1+2+\cdots+(k+1))^2
\end{aligned}$$

which is what we needed, and we are done.

5. *Find a formula for*

(a) $\displaystyle\sum_{i=1}^{n}(2i-1) = 1+3+5+7+\cdots+(2n-1)$

$$\begin{aligned}
\sum_{k=1}^{2n} k &= \sum_{k=1}^{n}(2k-1)+\sum_{k=1}^{n}2k \\
\frac{(2n)(2n+1)}{2} &= \sum_{k=1}^{n}(2k-1)+2\sum_{k=1}^{n}k \\
\sum_{k=1}^{n}(2k-1) &= 2n^2+n-2\frac{n(n+1)}{2} \\
&= 2n^2+n-(n^2+n) = n^2
\end{aligned}$$

(b) $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+7^2+\cdots+(2n-1)^2$

Solution Method I:

$$\begin{aligned}
\sum_{k=1}^{2n} k^2 &= \sum_{k=1}^{n}(2k-1)^2+\sum_{k=1}^{n}(2k)^2 \\
\sum_{k=1}^{n}(2k-1)^2 &= \frac{(2n)(2n+1)(4n+1)}{6}-4\sum_{k=1}^{n}k^2 \\
&= \frac{(2n)(2n+1)(4n+1)}{6}-\frac{4n(n+1)(2n+1)}{6} \\
&= \frac{2n(2n+1)(2n-1)}{6} = \frac{4n^3-n}{3}
\end{aligned}$$

Solution Method II:

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1)^2 &= \sum_{k=1}^{n}(4k^2-4k+1)\\
&= \sum_{k=1}^{n}4k^2-\sum_{k=1}^{n}4k+\sum_{k=1}^{n}1\\
&= 4\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+n\\
&= 4\left(\frac{n(n+1)(2n+1)}{6}\right)-4\left(\frac{n(n+1)}{2}\right)+n\\
&= \frac{2n(n+1)(2n+1)}{3}-2n^2-n\\
&= \frac{n(2n+1)(2n-1)}{3}=\frac{4n^3-n}{3}
\end{aligned}$$

6. *Use the given method to find:*

(a) $1^3+2^3+3^3+4^3+\cdots+n^3$

Following the example from the homework sheet we note that $(k+1)^4-k^4=4k^3+6k^2+4k+1$. Then proceeding as the example we would have:

$$\begin{aligned}
(n+1)^4-1 &= 4\sum_{k=1}^{n}n^3+6\sum_{k=1}^{n}n^2+4\sum_{k=1}^{n}n+n\\
4\sum_{k=1}^{n}n^3 &= (n+1)^4-1-6\frac{n(n+1)(2n+1)}{6}-4\frac{n(n+1)}{2}-n\\
4\sum_{k=1}^{n}n^3 &= (n+1)^4-1-n(n+1)(2n+1)-2n(n+1)-n\\
4\sum_{k=1}^{n}n^3 &= n^4+2n^3+n^2\\
\sum_{k=1}^{n}n^3 &= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

(b) $1^4+2^4+3^4+4^4+\cdots+n^4$

First, $(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$. So,

$$\begin{aligned}
(n+1)^5 - 1 &= 5\sum_{k=1}^{n} n^4 + 10\sum_{k=1}^{n} n^3 + 10\sum_{k=1}^{n} n^2 + 5\sum_{k=1}^{n} n + n \\
5\sum_{k=1}^{n} n^4 &= (n+1)^5 - 1 - 10\frac{n^2(n+1)^2}{4} - 10\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - n \\
5\sum_{k=1}^{n} n^4 &= n^5 + \frac{5n^4}{2} + \frac{5n^3}{3} - \frac{n}{6} \\
5\sum_{k=1}^{n} n^4 &= \frac{n(2n+1)(n+1)(3n^2+3n-1)}{6} \\
\sum_{k=1}^{n} n^4 &= \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30}
\end{aligned}$$

(c) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}$

For this one you need to realize that each term is of the form $\dfrac{1}{k\cdot(k+1)}$ and this can be rewritten as $\dfrac{1}{k\cdot(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Thus,

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} = \frac{n}{n+1}
\end{aligned}$$

This is a classic example of what is known as a *telescoping sum*.