MATH 6101-090

ASSIGNMENT 2 - SOLUTIONS

18-September-2006

1. *Let s be a nonzero rational number and t be irrational.* For the following problems recall that we know that the sum of two rationals is rational, the product of two rationals is rational and the quotient of two rationals is rational.

   (a) *Prove that $s - t$ is irrational.* Assume that $s - t$ is a rational number, r. Then $s - t = r$ means that $t = s - r$. Since s and r are rational numbers, $s - r$ is a rational number, which makes t a rational number. We were given that it is irrational, so the assumption that $s - t$ is rational leads to a contradiction and $s - t$ must be irrational.

   (b) *Prove that st is irrational.* Assume that $st = r$ is rational. Then $t = r/s$ is the quotient of two rational numbers and hence is rational. This contradicts the given condition that t is irrational, so we must conclude that st is irrational.

   (c) *Prove that s/t is irrational.* Just as above, assume that $s/t = r$ is rational. Then since $s \neq 0$, we know that $r \neq 0$ and we can solve for t. $t = s/r$. Since s and r are rational, that would make t rational, contradicting what is given. Thus, s/t is irrational.

2. *Although $\sqrt{2} + \sqrt{3}$ does not equal the square root of an integer, $\sqrt{27} + \sqrt{48}$ does.*

   (a) *What integer's square root equals $\sqrt{27} + \sqrt{48}$?*

   $$\sqrt{27} + \sqrt{48} = 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3} = \sqrt{147}.$$

   (b) *Find another different example like $\sqrt{27} + \sqrt{48}$.*

   $$\sqrt{272} + \sqrt{425} = \sqrt{1377}.$$

   (c) *Find every set of different positive integers p, q, and r all less than 100 such that p and q are not perfect squares and $\sqrt{p} + \sqrt{q} = \sqrt{r}$.*

   From above you should have noticed that p and q must be of the form k^2t where t is a square-free integer. So, let $p = n^2t$ and $q = m^2t$, where t is square-free.

   $$\sqrt{p} + \sqrt{q} = n\sqrt{t} + m\sqrt{t} = (n + m)\sqrt{t},$$

   which makes $r = (n + m)^2t$. So, we want p, q, and r to be distinct and less than 100. That means that $n \neq m$ and the smallest that we can take n and m is 1 and 2. Thus $r = (n + m)^2t = 9t < 100$ means that t cannot be greater than 11. So, let's look a table of possible values:

t	n	m	$p = n^2t$	$q = m^2t$	$r = (n+m)^2t$
2	1	2	2	8	18
2	1	3	2	18	32
2	1	4	2	32	50
2	1	5	2	50	72
2	1	6	2	72	98
2	2	3	8	18	50
2	2	4	8	32	72
2	2	5	8	50	98
2	3	4	18	32	98
3	1	2	3	12	27
3	1	3	3	27	48
3	1	4	3	48	75
3	2	3	12	27	75
5	1	2	5	20	45
5	1	3	5	45	80
6	1	2	6	24	54
6	1	3	6	54	96
7	1	2	7	28	63
10	1	2	10	40	90
11	1	2	11	44	99

3. *Suppose that p and q are positive integers and that $a = \frac{p}{q}$.* ***Explain*** *why long division of p by q results in the decimal representation of a.*
[Hint: It is enough to explain why the decimal $d = [D, d_1, d_2, d_3, \dots, d_k, \dots]$ produced by long division satisfies

$$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} \le a \le D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}$$

for all $k \in \mathbb{N}$.]

We saw that long division comes from the Euclidean algorithm and we have that

$$p = q \times \left(D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} \right) + \frac{r}{10^k}$$

where $0 \le r \le q$.

Thus, dividing the above by q, get that

$$a = \frac{p}{q} = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} + \frac{s}{10^k}, \text{ where } 0 \le s \le 1.$$

This says that a must be bigger than $D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k}$ which is the number we get when $s = 0$ and must be less than $D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}$, which is what we get when

$s = 1$. Thus,

$$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} \leq a \leq D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}$$

4. *Prove the following properties of any complex number z.*

   (a) $\mathrm{Re}[z] = \frac{z+\bar{z}}{2}$ Let $z = x + iy$, then $\mathrm{Re}[z] = x$ and $\bar{z} = x - iy$.

   $$\frac{z+\bar{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = \frac{2x}{2} = x = \mathrm{Re}[z],$$

   and we are done.

   (b) $\mathrm{Im}[z] = \frac{z-\bar{z}}{2i}$ Similarly,

   $$\frac{z-\bar{z}}{2i} = \frac{(x+iy)-(x-iy)}{2i} = \frac{2iy}{2i} = y = \mathrm{Im}[z].$$

5. *Suppose that $[r, \theta]$ is a point z in the complex plane, $x = r\cos\theta$ and $y = r\sin\theta$. If $r' = -r$ and $\theta' = \theta + \pi(2n+1)$, where n is an integer, prove that $[r', \theta']$ determines the same rectangular representation (x, y) of z.*

   The rectangular representation of (x, y) is $x = r\cos\theta$ and $y = r\sin\theta$. The rectangular representation from the polar coordinate $[r', \theta']$ is

   $$\begin{aligned}
   r'\cos\theta' &= (-r)\cos(\theta + (2n+1)\pi) \\
   &= (-r)(\cos\theta\cos((2n+1)\pi) - \sin\theta\sin((2n+1)\pi)) \\
   &= (-r)((\cos\theta)(-1)) = r\cos\theta = x \\
   r'\sin\theta' &= (-r)\sin(\theta + (2n+1)\pi) \\
   &= (-r)(\cos\theta\sin((2n+1)\pi) + \sin\theta\cos((2n+1)\pi)) \\
   &= (-r)((\sin\theta)(-1)) = r\sin\theta = y
   \end{aligned}$$

6. *Track the solution set in the complex plane of the quadratic equation $x^2 + bx + 2 = 0$ as the value of the real coefficient b varies.*

   The discriminant of the quadratic is $b^2 - 8$. This will be negative for $0 \leq |b| < \sqrt{8}$ and positive for $|b| > \sqrt{8}$. Thus it will have complex roots for b in the first interval, a single real root when $|b| = \sqrt{8}$ and two distinct real roots for $|b| > \sqrt{8}$. All of the roots are of the form $\frac{-b\pm\sqrt{b^2-8}}{2}$. Looking at a table of roots, we see

b	b^2-8	r_1	r_2
-3	1	2	1
$-\sqrt{8}$	0	$\sqrt{2}$	$\sqrt{2}$
-2	-4	$1+i$	$1-i$
-1	-7	$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$
0	-8	$i\sqrt{2}$	$-i\sqrt{2}$
1	-7	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
2	-4	$-1+i$	$-1-i$
$\sqrt{8}$	0	$-\sqrt{2}$	$-\sqrt{2}$
3	1	-1	-2

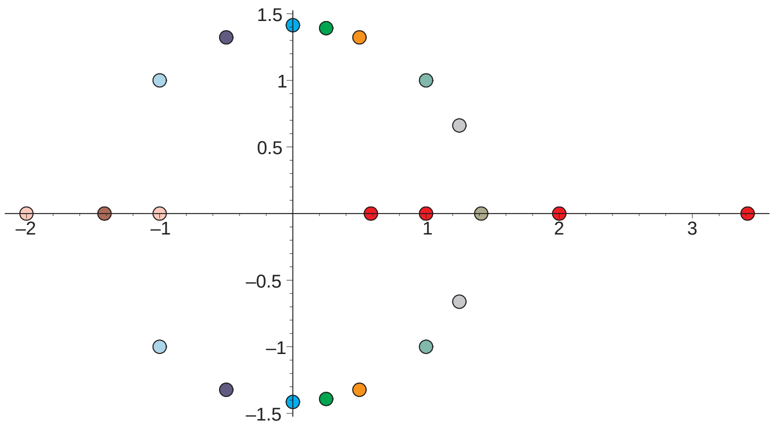

Figure 1: Tracking roots in the complex plane

Note that the real roots cover the positive real axis for $b < \sqrt{8}$, since as $b \mapsto -\infty$, one root goes to 0 and the other root goes to $-\infty$. Likewise for $b > \sqrt{8}$, the roots cover the negative real axis. When $|b| < \sqrt{8}$, the roots all lie on the complex circle centered at 0 of radius $\sqrt{2}$.

7. *Find the fifth roots of unity. Find the fifth roots of i. Plot them in the complex plane. How are they related?*

The fifth roots of unity are given by $\{1, \omega, \omega^2, \omega^3, \omega^4\}$ where

$$\begin{aligned}
\omega &= \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) = \frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}; \\
\omega^2 &= \cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5}) = \frac{-1-\sqrt{5}}{4} + i\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}; \\
\omega^3 &= \cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5}) = \frac{-1-\sqrt{5}}{4} - i\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}; \\
\omega^4 &= \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) = \frac{-1+\sqrt{5}}{4} - i\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}
\end{aligned}$$

In polar coordinates these are:

$$\omega = \left[1, \frac{2\pi}{5}\right]; \quad \omega^2 = \left[1, \frac{4\pi}{5}\right]; \quad \omega^3 = \left[1, \frac{6\pi}{5}\right]; \quad \omega^4 = \left[1, \frac{8\pi}{5}\right]; \quad \omega^5 = [1, 0].$$

The fifth roots of i can be found similarly to the fifth roots of 1. Remember that i can be represented in the complex plane as $(0,1)$ or in polar coordinates as $[1, \pi/2]$. As in the class notes, we will have $r^5 = 1$ which makes $r = 1$ and $5\theta = \frac{\pi}{2} + 2\pi k$. Thus, we have five different angles that will satisfy this:

$$\theta_1 = \frac{\pi}{10}, \;\; \theta_2 = \frac{\pi}{10} + \frac{2\pi}{5} = \frac{\pi}{2}, \;\; \theta_3 = \frac{\pi}{10} + \frac{4\pi}{5} = \frac{9\pi}{10}, \;\; \theta_4 = \frac{\pi}{10} + \frac{6\pi}{5} = \frac{13\pi}{10}, \;\; \theta_5 = \frac{\pi}{10} + \frac{8\pi}{5} = \frac{17\pi}{10}$$

and the fifth roots of i in polar coordinates are:

$$\xi = \left[1, \frac{\pi}{10}\right]; \quad \xi^2 = \left[1, \frac{\pi}{2}\right]; \quad \xi^3 = \left[1, \frac{9\pi}{10}\right]; \quad \xi^4 = \left[1, \frac{13\pi}{10}\right]; \quad \xi^5 = \left[1, \frac{17\pi}{10}\right].$$

You should note that the fifth roots of i can be obtained from the fifth roots of 1 by rotation by $\pi/2$ radians (or 90°).

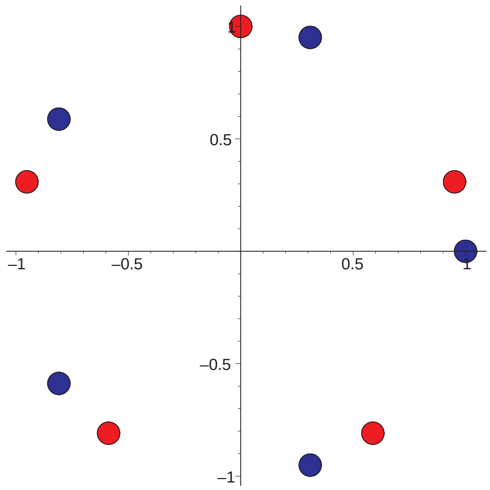

Figure 2: Red dots - fifth roots of i; Blue dots - fifth roots of 1. Don't agitate the dots!

8. *Find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.*

   (a) $z_1 = -2 + 8i$, $z_2 = -2 - 8i$.

$$\begin{aligned}
z_1 + z_2 &= (-2+8i) + (-2-8i) = -4 \\
z_1 - z_2 &= (-2+8i) - (-2-8i) = 16i \\
z_1 z_2 &= (-2+8i)(-2-8i) = 4 + 64 = 68 \\
\frac{z_1}{z_2} &= \frac{-2+8i}{-2-8i} = \frac{(-2+8i)(-2+8i)}{(-2-8i)(-2+8i)} = -\frac{60}{68} - \frac{32}{68}i = \frac{1}{17}(-15-8i)
\end{aligned}$$

   (b) $z_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $z_2 = (1, 0)$.

$$\begin{aligned}
z_1 + z_2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + (1,0) = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \\
z_1 - z_2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) - (1,0) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
z_1 z_2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot (1,0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
\frac{z_1}{z_2} &= \frac{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}{(1,0)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)
\end{aligned}$$

(c) $z_1 = [3, 225°] = [3, \frac{5\pi}{4}]$, $z_2 = 7(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$. $z_1 = 3(\cos 225° + i\sin 225°) = -\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i$ and $z_2 = 7(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = \frac{7}{2} - \frac{7\sqrt{3}}{2}i$ and has polar coordinate $[7, \frac{5\pi}{3}]$.

$$\begin{aligned}
z_1 + z_2 &= \left(-\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i\right) + \left(\frac{7}{2} - \frac{7\sqrt{3}}{2}i\right) = \frac{7-3\sqrt{2}}{2} - \frac{7\sqrt{3}+3\sqrt{2}}{2} \\
z_1 - z_2 &= \left(-\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i\right) - \left(\frac{7}{2} - \frac{7\sqrt{3}}{2}i\right) = -\frac{7+3\sqrt{2}}{2} + \frac{7\sqrt{3}-3\sqrt{2}}{2} \\
z_1 z_2 &= [3, \frac{5\pi}{4}][7, \frac{5\pi}{3}] = [21, \frac{35\pi}{12}] = [21, \frac{11\pi}{12}] \\
\frac{z_1}{z_2} &= \frac{[3, \frac{5\pi}{4}]}{[7, \frac{5\pi}{3}]} = \left[\frac{3}{7}, -\frac{5\pi}{12}\right]
\end{aligned}$$

(d) *z_1 and z_2 are the solutions to $x^2 + x + 1 = 0$.*

There is another way of finding the sum and the product of the roots. The sum of the roots is the negative of the coefficient of x and the product of the roots is the constant term. Thus, $z_1 + z_2 = -1$ and $z_1 \cdot z_2 = 1$. Note also that, since the roots appear in conjugate pairs, z_1 and z_2 are conjugates of each other, *i.e.*, $\bar{z}_1 = z_2$ and $\bar{z}_2 = z_1$. Now, let's find the roots and then you can check the above.

$$z_1 = \frac{-1+\sqrt{1-4}}{2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

$$\begin{aligned}
z_1 + z_2 &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -1 \\
z_1 - z_2 &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = i\sqrt{3} \\
z_1 z_2 &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{4} + \frac{3}{4}\right) + i\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) = 1 \\
\frac{z_1}{z_2} &= \frac{(-\frac{1}{2} + i\frac{\sqrt{3}}{2})}{(-\frac{1}{2} - i\frac{\sqrt{3}}{2})} = \frac{(\frac{1}{4} - \frac{3}{4}) - i\frac{\sqrt{3}}{2}}{1} = -\frac{1}{2}(1 + i\sqrt{3}) = z_2
\end{aligned}$$

Note, that this last one should not be surprising given the information we had:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1^2}{z_1 z_2} \\
&= z_1^2 \text{ since } z_1 z_2 = 1 \\
z_1 + z_2 &= -1 \text{ so} \\
z_1^2 + z_1 z_2 &= -z_1 \\
\frac{z_1}{z_2} &= z_1^2 = -z_1 - z_1 z_2 = -z_1 - 1 = z_2
\end{aligned}$$