# *ASSIGNMENT 7*
# *SOLUTIONS*

04-November-2006

1. *Prove that*

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$$

*for all* $n \geq 2$.

Check this for $n = 3$ and we have

$$\begin{aligned}\prod_{k=2}^{3}\left(1-\frac{1}{k^2}\right)&=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\\&=\frac{3}{4}\cdot\frac{8}{9}=\frac{2}{3}\\&=\frac{3+1}{2\cdot 3}\end{aligned}$$

Thus, this is true for the first case. Now, assume that the formula is true for $n$ and we need to show that it is true for $n+1$. Thus, we know that

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}.$$

We need to show that

$$\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)=\frac{n+2}{2n+2}$$

$$\begin{aligned}\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)&=\left(1-\frac{1}{(n+1)^2}\right)\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)\\&=\left(1-\frac{1}{(n+1)^2}\right)\frac{n+1}{2n}\\&=\frac{n+1}{2n}-\frac{1}{2n(n+1)}=\frac{(n+1)^2-1}{2n(n+1)}\\&=\frac{n+2}{2(n+1)}\end{aligned}$$

which is what we were to show.

2. *Suppose that* $f\colon X \to Y$ *is onto and* $A \subseteq Y$. *Prove that*

$$f(f^{-1}(A)) = A.$$

We need to show that $f(f^{-1}(A)) \subseteq A$ and $A \subseteq f(f^{-1}(A))$.

Let $a \in A$. Since $f$ is onto, there is an $x \in X$ so that $f(x) = a$. Since $f(x) = a \in A$, $x \in f^{-1}(A)$ which makes $a \in f(f^{-1}(A))$ and we have shown that $A \subseteq f(f^{-1}(A))$.

Let $y \in f(f^{-1}(A))$. Then $y = f(x)$ where $x \in f^{-1}(A)$. By definition $x \in f^{-1}(A)$ if and only if $f(x) \in A$. Thus, $y = f(x) \in A$ and $f(f^{-1}(A)) \subseteq A$.

Therefore, we have shown that $A = f(f^{-1}(A))$.

3. *Decide if the following statements are* ***true*** *or* ***false****. If* ***false****, give an example showing the statement is* ***false****.*

   (a) *If $f\colon \mathbb{N} \to \mathbb{N}$ is one-to-one, then f is onto.*

   This is false. Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$. This function is clearly one-to-one, but it is not onto since there is no element in $\mathbb{N}$ that is sent to 1.

   (b) *If $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, then there is an inverse function $f^{-1}$.*

   This is true.

   (c) *If $A \subset \mathbb{R}$ is bounded above, then there is an element $a \in A$ that is a least upper bound for $A$.*

   This is false. While it is true that the set must have a least upper bound, the least upper bound does not have to belong to $A$. Let $A = (-\infty, 2)$. Then 2 is the least upper bound for $A$, but $2 \notin A$.

   (d) *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon [0, +\infty) \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.*

   It is true that $f(g(x)) = x$ since we are starting in the positive reals. However, it is not true that $g(f(x)) = x$ for all real numbers: $g(f(-2)) = \sqrt{4} = 2 \neq -2$. This statement is false.

   (e) *If $x^2 < y^2$, then $x < y$.*

   This is false. Consider $x = -2$ and $y = -3$. It is true that $x^2 = 4 < 9 = y^2$, but $y = -3 < -2 = x$.

4. *Give examples of the following phenomena.*

   (a) *A function $f\colon \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.*

   There are a number of functions that will work, such as $f(x) = \arctan(x)$ or $f(x) = e^x$.

   (b) *A function $f\colon \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.*

   Playing off of the previous function from $\mathbb{N}$ to $\mathbb{N}$, let the function $f$ be given by

$$f\colon x \mapsto \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

   Now, $f(1) = 1 = f(2)$ and the function is otherwise clearly onto.

   (c) *A function $f\colon \mathbb{R} \to \mathbb{R}$ and sets $A, B \subset \mathbb{R}$ such that $f(A \cap B) \not\supseteq f(A) \cap f(B)$.*

   For this one let $f\colon x \mapsto x^2$ and let $A = (-\infty, 0)$ and $B = (0, +\infty)$. Then, $A \cap B = \emptyset$ so $f(A \cap B) = \emptyset$. However, $f(A) = (0, +\infty) = f(B)$, so $f(A) \cap f(B) = (0, +\infty) \not\subseteq f(A \cap B)$.

(d) *A sequence of intervals* $J_n = (a_n, b_n)$ *where* $a_n < b_n$ *for all* $n$ *and* $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$, *but*

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

Let $J_n = \left(0, \frac{1}{n}\right)$, for $n = 1, 2, 3, \ldots$. Then, $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$, but

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

(e) *A sequence of intervals* $J_n = (a_n, b_n)$ *where*

$$a_1 < a_2 < a_3 < \cdots < a_n < \cdots < b_n < \cdots < b_2 < b_1$$

*but*

$$\bigcap_{n=1}^{\infty} J_n \neq \emptyset.$$

We will take an example like the previous one. Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ so the first condition is satisfied, but $0 \in J_n$ for all $n$ which makes

$$\bigcap_{n=1}^{\infty} J_n = \{0\} \neq \emptyset.$$

5. *Suppose* $a, b, x, y > 0$ *and* $\frac{a}{b} < \frac{x}{y}$. *Prove that* $\frac{a}{b} < \frac{a+x}{b+y}$.

Since $\frac{a}{b} < \frac{x}{y}$ we have

$$\begin{aligned} \frac{a}{b} &< \frac{x}{y} \\ ay &< bx \\ ab + ay &< ab + bx \\ a(b+y) &< b(a+x) \\ \frac{a}{b} &< \frac{a+x}{b+y} \end{aligned}$$