ASSIGNMENT 7

30-October-2006

1. Prove that
$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$$
for all $n \geq 2$.

2. Suppose that $f\colon X \to Y$ is onto and $A \subseteq Y$. Prove that
$$f(f^{-1}(A)) = A.$$
[HINT: Use elements of the sets.]

3. Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing the statement is FALSE.
   (a) If $f\colon \mathbb{N} \to \mathbb{N}$ is one-to-one, then f is onto.
   (b) If $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, then there is an inverse function f^{-1}.
   (c) If $A \subset \mathbb{R}$ is bounded above, then there is an element $a \in A$ that is a least upper bound for A.
   (d) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon [0, +\infty) \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.
   (e) If $x^2 < y^2$, then $x < y$.

4. Give examples of the following phenomena.
   (a) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.
   (b) A function $f\colon \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.
   (c) A function $f\colon \mathbb{R} \to \mathbb{R}$ and sets $A, B \subset \mathbb{R}$ such that $f(A \cap B) \not\supseteq f(A) \cap f(B)$.
   (d) A sequence of intervals $J_n = (a_n, b_n)$ where $a_n < b_n$ for all n and $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$, but
   $$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$
   (e) A sequence of intervals $J_n = (a_n, b_n)$ where
   $$a_1 < a_2 < a_3 < \cdots < a_n < \cdots < b_n < \cdots < b_2 < b_1$$
   but
   $$\bigcap_{n=1}^{\infty} J_n \neq \emptyset.$$

5. Suppose $a, b, x, y > 0$ and $\frac{a}{b} < \frac{x}{y}$. Prove that $\frac{a}{b} < \frac{a+x}{b+y}$.