ASSIGNMENT 9

13-November-2006

1. Let $f\colon (a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$. Show that if $f(r) = 0$ for each rational number $r \in (a,b)$, then $f(x) = 0$ for all $x \in (a,b)$.

2. Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$, so that $f(r) = g(r)$ for each rational number $r \in (a,b)$. Prove that $f(x) = g(x)$ for all $x \in (a,b)$.

3. Define the function f by
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{otherwise.} \end{cases}$$
Show that f is discontinuous at every $x \in \mathbb{R}$.

4. Define the function h by
$$h(x) = \begin{cases} x & \text{if } x \text{ is rational;} \\ 0 & \text{otherwise.} \end{cases}$$
Show that h is continuous at $x = 0$ and at no other point.

5. For each rational number x, write x as $\frac{p}{q}$ where p and q are integers with no common factors and $q > 0$. Define the function g by
$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x \text{ is rational;} \\ 0 & \text{otherwise.} \end{cases}$$
Thus, $g(x) = 1$ for all integers, $g(\frac{1}{2}) = g(-\frac{1}{2}) = g(\frac{3}{2}) = \frac{1}{2}$. Show that g is continuous at each irrational and discontinuous at each rational.

6. Let f and g be continuous functions on $[a,b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$.

7. Prove that $x2^x = 1$ for some $x \in (0,1)$.