Problems for the Final Exam
MATH 6101 - Fall 2008

1. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all natural numbers $n \in \mathbb{N}$.

2. Prove that $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1) = 3n^2$ for all natural numbers $n \in \mathbb{N}$.

3. The decimal number $0.a_1 \ldots a_s a_{s+1} \ldots a_t a_{s+1} \ldots a_t a_{s+1} \ldots$ is denoted by $0.a_1 \ldots a_s \overline{a_{s+1} \ldots a_t}$. Prove that every decimal number of the form $N.a_1 \ldots a_s \overline{a_{s+1} \ldots a_t}$ can be written as a rational number.

4. Use the formula
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
to prove that
$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$
Use the series for arctangent discovered by Gregory
$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots,$$
through the x^7 term and a calculator to estimate the above value for π.

5. Does the sequence $\left\{\frac{1}{\sqrt[3]{n+1} - \sqrt[3]{n}}\right\}$ converge or diverge?

6. (a) Give an example of a bounded sequence or real numbers that does not converge.
   (b) Give an example of a bounded sequence of rational numbers that converges to an irrational number.
   (c) Give an example of a bounded sequence of irrational numbers that converges to a rational number.
   (d) Give an example of a bounded sequence of irrational numbers that converges to an irrational number.

7. Let $a_1 = 1$ and $a_{n+1} = \dfrac{a_n^2 + 2}{2a_n}$ for $n \geq 1$. Assume that $\{a_n\}$ converges and find the limit.

8. Suppose that $a_0 > 0$, $r > 0$ and $a_{n+1} = r\left(a_n + \dfrac{1}{a_n}\right)$. For which values of r does $\{a_n\}$ converge and for which does it diverge? Find the value of the limit when it exists.

9. Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

10. Let $\{a_n\}$ be a sequence of nonzero real numbers such that the sequence $\{\frac{a_{n+1}}{a_n}\}$ is a constant sequence. Show that $\sum a_n$ is a geometric series.

11. Suppose that $a_0 = 1$ and $a_{n+1} = a_n + \dfrac{1}{n^2 a_n}$. Does $\{a_n\}$ converge or diverge. Hint: How can you think of this as a series?

12. Suppose that $a_n \geq 0$ and $\sum a_n$ converges. Prove that $\sum a_n^2$ also converges.

13. Suppose that $\sum a_n$ and $\sum b_n$ both converge. Prove that if $a_n \geq 0$ and $b_n \geq 0$, then $\sum a_n b_n$ also converges.

14. Suppose $\sum a_n$ and $\sum b_n$ both diverge. Does $\sum(a_n + b_n)$ necessarily diverge? If yes, why? If no, provide an example.

15. Prove that the power series $\sum n^n x^n$ converges only for $x = 0$.

16. Prove that the power series $\sum \dfrac{x^n}{n^n}$ converges for all x.

17. Suppose that there exist polynomials $P(x)$ and $Q(x)$ so that $c_n = \dfrac{P(n)}{Q(n)}$. Prove that the power series $\sum c_n x^n$ has radius of convergence $\rho = 1$.

18. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there is an x between a and b such that $f(x) = 0$.

19. Suppose that f is continuous on $[0, 2]$ and the $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ so that $|y - x| = 1$ and $f(x) = f(y)$.

20. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

    Evaluate the following

    (a) $\lim_{x \to 0} f(x)$

    (b) $\lim_{x \to 5} f(x)$

    (c) $\lim_{x \to \sqrt{2}} f(x)$