MATH 6101
Fall 2008

Euler and Trigonometric Series

Sums of Series

What do we know about sums of series?

How many series sums do we know? can we find?

Examples:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Sums of Series

Claim 1: $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$

$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}} \right) = \sum_{n=1}^{\infty} \frac{n}{2^n} - \sum_{n=1}^{\infty} \frac{n+1}{2^{n+1}}$$

$$= \frac{1}{2} + \sum_{k=2}^{\infty} \frac{k}{2^k} - \sum_{k=2}^{\infty} \frac{k}{2^k}$$

$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$$

Sums of Series

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{k}{2^{k}}$$

$$\sum_{k=1}^{\infty} \frac{k}{2^{k}} = 2$$

Sums of Series

Claim 2: $\sum_{n=1}^{\infty}\frac{n^2}{2^n}=6$

$$\sum_{n=1}^{\infty}\frac{n^2-1}{2^{n+1}}=\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}-\frac{n^2+1}{2^{n+1}}\right)=\sum_{n=1}^{\infty}\frac{n^2}{2^n}-\sum_{n=1}^{\infty}\frac{n^2+1}{2^{n+1}}+2\sum_{n=1}^{\infty}\frac{n}{2^n}-\sum_{n=1}^{\infty}\frac{2n}{2^n}$$

$$=\sum_{n=1}^{\infty}\frac{n^2}{2^n}-\sum_{n=1}^{\infty}\frac{n^2+2n+1}{2^{n+1}}+4$$

$$=4+\sum_{n=1}^{\infty}\frac{n^2}{2^n}-\sum_{n=1}^{\infty}\frac{(n+1)^2}{2^{n+1}}$$

$$\sum_{n=1}^{\infty}\frac{n^2-1}{2^{n+1}}=4+\frac{1}{2}+\sum_{k=2}^{\infty}\frac{k^2}{2^k}-\sum_{k=2}^{\infty}\frac{k^2}{2^k}=\frac{5}{2}$$

Sums of Series

$$\frac{5}{2} = \sum_{n=1}^{\infty} \frac{n^2 - 1}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{n^2}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2}{2^n} - \sum_{n=2}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2}{2^n} - \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

Sums of Series

In a like manner we can show that

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n} = 26$$

On the other hand we know that

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

Sums of Series

How about $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

First, does it converge?

We know that $2n^2 \geq n(n+1)$ so $\frac{2}{n(n+1)} \geq \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2$$

Therefore, it does converges. But to what?

Sums of Series

Jakob Bernoulli considered it and failed to find it. So did Mengoli and Leibniz. Finding the sum became known as the *Basel Problem*.

We will look at how Euler solved the problem.

Euler's Sum

Theorem [Euler, 1735]: $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This was a huge surprise, as no one expected π to appear in the sum at all!!

Euler's Proof: Let $p(x)$ be a polynomial of degree n with the following properties

1. $p(x)$ has non-zero roots $a_1, a_2, \ldots, a_n$.
2. $p(0) = 1$

Then we can write:

Euler's Sum

$$p(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\cdots\left(1 - \frac{x}{a_n}\right)$$

Euler claims that "*what holds for a finite polynomial holds for an infinite polynomial*".

He considers the infinite polynomial.

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

which is an infinite polynomial with $p(0) = 1$.

Euler's Sum

Euler knew an infinite polynomial for sin(x).

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!} + \cdots$$

So Euler noticed that $xp(x)$

$$xp(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x)$$

From the properties of the sine function we know that this infinite polynomial has zeros at

$$x = \pm k\pi \quad \text{for } k = 1,2,3,\ldots$$

Euler's Sum

From his claim above, Euler could write this polynomial as an infinite product.

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right) \times \cdots$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right) \times \cdots$$

Euler's Sum

Now, we can multiply out the product and collect coefficients of like powers.

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots=1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right)x^2+\cdots$$

Equating the coefficients of x^2, we get

$$-\frac{1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right)$$

$$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Euler's Sum

Euler did not stop there. He had a good thing going.

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$$

In 1744 he obtained

$$\sum_{n=1}^{\infty}\frac{1}{n^{26}}=\frac{2^{24}76977927\pi^{26}}{27!}$$

Euler's Trigonometric Sums

Euler took what we knew for real numbers and pushed it to complex numbers.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and replace x by the complex number

$$z = a(\cos\varphi + i\sin\varphi)$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{1-a(\cos\varphi + i\sin\varphi)} = 1 + [a(\cos\varphi + i\sin\varphi)] + [a(\cos\varphi + i\sin\varphi)]^2 + [a(\cos\varphi + i\sin\varphi)]^3 + \cdots$$

Euler's Trigonometric Sums

Euler chose to look at the case $a = -1$. Proceed by rationalizing the denominator

$$\frac{1}{1+(\cos\varphi+i\sin\varphi)} = \frac{1}{1+(\cos\varphi+i\sin\varphi)}\bullet\frac{1+\cos\varphi-i\sin\varphi}{1+\cos\varphi-i\sin\varphi}$$

$$= \frac{1+\cos\varphi-i\sin\varphi}{(1+\cos\varphi)^2+\sin^2\varphi} = \frac{1+\cos\varphi-i\sin\varphi}{2(1+\cos\varphi)}$$

$$= \frac{1}{2} - \frac{i\sin\varphi}{2(1+\cos\varphi)}$$

Euler's Trigonometric Sums

On the right hand side, use De Moivre's Theorem: $(\cos\varphi + i\sin\varphi)^k = \cos k\varphi + i\sin k\varphi$

$$1-(\cos\varphi+i\sin\varphi)+(\cos\varphi+i\sin\varphi)^2-(\cos\varphi+i\sin\varphi)^3+\cdots$$
$$=1-(\cos\varphi+i\sin\varphi)+(\cos 2\varphi+i\sin 2\varphi)-(\cos 3\varphi+i\sin 3\varphi)+\cdots$$
$$=\left(1-\cos\varphi+\cos 2\varphi-\cos 3\varphi+\cdots\right)-i(\sin\varphi-\sin 2\varphi+\sin 3\varphi+\cdots)$$

Equating the real parts of the equation gives:

$$\frac{1}{2}=1-\cos\varphi+\cos 2\varphi-\cos 3\varphi+\cos 4\varphi-\cos 5\varphi+\cdots$$

$$\cos\varphi-\cos 2\varphi+\cos 3\varphi-\cos 4\varphi+\cos 5\varphi-\cdots=\frac{1}{2}$$

Euler's Trigonometric Sums

Integrate with respect to φ.

$$\sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \frac{1}{5}\sin 5\varphi - \cdots = \frac{\varphi}{2} + C$$

Since sin(0) = 0, we have $C = 0$ and

$$\sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \frac{1}{5}\sin 5\varphi - \cdots = \frac{\varphi}{2}$$

Integrate again with respect to φ.

$$\cos\varphi - \frac{1}{4}\cos 2\varphi + \frac{1}{9}\cos 3\varphi - \frac{1}{16}\cos 4\varphi + \frac{1}{25}\cos 5\varphi - \cdots = -\frac{\varphi^2}{4} + C$$

Since cos(0) = 1, we have

Euler's Trigonometric Sums

$$C = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

What is C? Is it related to the previous sum?

Use the fact that $\cos(2\pi+\alpha) = \cos \alpha$, make the substitution $\varphi = \pi/2$ and we get

$$C - \frac{(\pi/2)^2}{4} = \cos\frac{\pi}{2} - \frac{1}{4}\cos\pi + \frac{1}{9}\cos\frac{3\pi}{2} - \frac{1}{16}\cos 4\pi + \frac{1}{25}\cos\frac{5\pi}{2} - \cdots$$

$$= 0 + \frac{1}{4} + 0 - \frac{1}{16} + 0 + \frac{1}{36} - \cdots$$

$$= \frac{1}{4}\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right) = \frac{C}{4}$$

Euler's Trigonometric Sums

$$C - \frac{(\pi/2)^2}{4} = \frac{C}{4}$$

$$\frac{3C}{4} = \frac{(\pi/2)^2}{4}$$

$$C = \frac{\pi^2}{12}$$

You can use this process to show the previous sum, as well.

Euler's Trigonometric Sums

Note that from our earlier results, for $x \varepsilon (\pi,-\pi)$.

$$x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \cdots\right)$$

$$x^2 = 4\left(\frac{\pi^2}{12} - \cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + \frac{1}{25}\cos 5x - \cdots\right)$$

This ability to represent functions as trigonometric series languished until the work of Fourier in 1807 when he showed that these trigonometric series could be used to model the propagation of heat through a material.

Euler's Trigonometric Sums

Note:

$$x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \cdots\right)$$

So

$$\frac{\pi}{3} = 2\left(\sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{3}\right) - \frac{1}{4}\sin\left(\frac{4\pi}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{3}\right) - \frac{1}{6}\sin\left(\frac{6\pi}{3}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{3}\right) - \cdots\right)$$

$$\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{3}0 - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{5}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{6}0 + \frac{1}{7}\frac{\sqrt{3}}{2} - \cdots\right)$$

$$\frac{\pi}{3} = \sqrt{3} - \sqrt{3}\frac{1}{2} + \sqrt{3}\frac{1}{4} - \sqrt{3}\frac{1}{5} + \sqrt{3}\frac{1}{7} - \cdots$$

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \cdots$$

Your Turn

Show:

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots$$