MATH 6101
Fall 2008

The Cauchy Property

$+\infty$ and $-\infty$

1) They are ***not*** real numbers and do ***not*** necessarily obey the rules of arithmetic for real numbers.
2) We often act as if they do.
3) We need guidelines.

Add $+\infty$ and $-\infty$ to **R** and extend the ordering by

$$-\infty < a < +\infty$$

for every real number $a \in \mathbf{R} \cup \{+\infty, -\infty\}$.

$+\infty$ and $-\infty$

If $a \in \mathbf{R}$ then we define the following

1) $a + \infty = +\infty$
2) $a - \infty = -\infty$
3) If $a > 0$, then $a \times \infty = \infty$ and $a \times -\infty = -\infty$
4) If $a < 0$, then $a \times \infty = -\infty$ and $a \times -\infty = +\infty$

We may adopt the following conventions:

$$a/\infty = 0 \text{ and } a/(-\infty) = 0$$

Limits of Sequences

Limit of $\{a_n\}$ exists IFF we can compute L.

Will this always work?

Can we always find the limit?

Do we have to be able to find the limit as a number?

Theorem

Theorem (last lecture): *Every convergent sequence is bounded.*

Is the converse true?

Is it true that every bounded sequence converges?

Find a proof or a counterexample.

Definitions

A sequence $\{a_n\}$ is ***increasing*** if $a_n \le a_{n+1}$ for every n.

A sequence $\{a_n\}$ is ***decreasing*** if $a_n \ge a_{n+1}$ for every n.

A sequence is *monotone* (*monotonic*) if it is either increasing or decreasing.

Examples

1) Find an example of an increasing sequence.
2) Find an example of a decreasing sequence.
3) Find an example of a sequence that is not monotonic.

Increasing Sequences

Decreasing Sequences

Non-monotonic Sequences

Monotone Convergence Theorem

Theorem: *Every bounded monotonic sequence converges.*

Proof:

Let $\{a_n\}$ be a bounded increasing sequence and let $S = \{a_n \mid n \in N\}$. Since the sequence is bounded, $a_n < M$ for some real number M and for all n.

Therefore S is bounded and has a least upper bound. Let u =lub S and let $\varepsilon > 0$.

Theorem

Proof:

Since u=lub S and $\varepsilon > 0$, $u - \varepsilon$ is ***not*** an upper bound for S. Thus there is an integer K so that $a_K > u - \varepsilon$. Since $\{a_n\}$ is increasing then for all $n > K$, $a_n \geq a_N$ and for all $n > K$

$$u - \varepsilon < a_n \leq u.$$

Thus, $|\, a_n - u| < \varepsilon$ for all $n > K$ and $\lim a_n = u =$ lub S.

Consequences

1) The decimal representation of a real number converges.

$$m < m.d_1d_2d_3d_4\ldots = m + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \le m+1$$

Let $a_n = m.d_1d_2d_3d_4\ldots d_n$. Then $a_n \le a_{n+1}$ so $\{a_n\}$ is increasing.

2) Let $a_0 = 1$ and $a_{n+1} = 1/(1+ a_n)$

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Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1+ \sqrt{a_n}$.
Does it converge? Is it monotone?
$a_0 = 1 \quad a_1 = 1+ \sqrt{a_0}) = 2$
$a_2 = 1+ \sqrt{a_1}) = 1+ \sqrt{2} \approx 2.4142\ldots$
$a_3 = 1+ \sqrt{a_2} = 1 + \sqrt{2.4142\ldots} \approx 2.55377\ldots$

Prove it is increasing by induction on *n*.

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Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1+ \sqrt{a_n}$.
Converges by Monotone Convergence Theorem. To what does it converge?
Assume: $\lim_{n\to\infty} a_n = L$
$a_{n+1} = 1+ \sqrt{a_n}$
$\lim_{n\to\infty} a_{n+1} = 1+ \lim_{n\to\infty} \sqrt{a_n}$
$L = 1+ \sqrt{(\lim_{n\to\infty} a_n)}$
$L = 1 + \sqrt{L}$
$(L - 1)^2 = L$ so $L^2 - 3L + 1 = 0$
$L = (3 \pm \sqrt{(9 - 4)})/2 = (3 \pm \sqrt{5})/2$
Which one is it? It cannot be both. Why?

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Theorem

Theorem: Let $\{a_n\}$ be a sequence of real numbers.

(i) If $\{a_n\}$ is an unbounded monotonically increasing sequence, then $\lim a_n = +\infty$.

(ii) If $\{a_n\}$ is an unbounded monotonically decreasing sequence, then $\lim a_n = -\infty$.

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Theorem

Theorem: Suppose that $\{a_n\}$ is a monotone increasing sequence and $\{b_n\}$ is a monotone decreasing sequence such that

$$a_n \leq b_n \text{ for all n} = 0,1,2,\ldots$$

and

$$\{a_n - b_n\} \to 0$$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

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Theorem

Theorem: *Every sequence contains a monotone subsequence.*

Proof: Let $\{a_n\}$ be a sequence. We say that a term a_n is *dominating* if $a_n > a_m$ for all $m > n$.

Claim: Every sequence contains an infinite number or a finite number of dominating terms. (Note: finite could be 0.)

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Theorem

Proof (continued):

(i) Assume $\{a_n\}$ has an infinite number of dominating terms. Call these $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$ where $n_0 < n_1 < n_2 < \ldots$. By definition

$$a_{n_0} > a_{n_1} > a_{n_2} > \ldots$$

which is the monotone subsequence

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Theorem

Proof (continued):

(ii) Assume $\{a_n\}$ has a finite number of dominating terms. Thus, there is an m so that for every $n > m$, a_n is not dominating.

That means that for each $n > m$ there exists a $k > n$ so that $a_n \leq a_k$. Let $n_0 = m$. By the above there is a $n_1 > n_0$ so that $a_{n_0} \leq a_{n_1}$. Since $n_1 > n_0$ then there is $n_2 > n_1$ so that $a_{n_1} \leq a_{n_2}$. This gives

$$a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \ldots$$

which is the required monotone subsequence.

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Bolzano-Weierstrauss Theorem

Theorem: *Every bounded sequence has a convergent subsequence.*

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The Cauchy Property

Definition 1: A sequence $\{a_n\}$ is said to have the Cauchy property if for every $\varepsilon > 0$ there is an index K so that

$$| a_{n+m} - a_n| < \varepsilon$$

for all $n \geq K$ and $m = 1,2,3,\ldots$

[Note: equivalent statement –

$\{a_{n+m}\}^{\infty}_{m=0} \subset (a_n - \varepsilon, a_n + \varepsilon)$ for all $n \geq K$.]

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The Cauchy Property

Definition 2: A sequence $\{a_n\}$ is said to have the Cauchy property if for every $\varepsilon > 0$ there is an index K so that if $n,m > K$ then

$$| a_m - a_n| < \varepsilon.$$

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Definitions

Let $\{a_n\}$ be bounded – convergent or not, it does not matter.

Limiting behavior of $\{a_n\}$ depends only on the *tails* of the sequence, $\{a_n \mid n > N\}$.

Let $u_N = \text{glb}\{a_n \mid n > N\}$

Let $v_N = \text{lub}\{a_n \mid n > N\}$

FACT: If lim a_n exists, then it lies in $[u_N, v_N]$.

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Definitions

As N increases, the sets $\{a_n \mid n > N\}$ get smaller. Thus,

$u_1 \le u_2 \le u_3 \le \ldots$ and $v_1 \ge v_2 \ge v_3 \ge \ldots$

Let

$u = \lim_{N\to\infty} u_N$ and $v = \lim_{n\to\infty} v_N$

Both exist – Why?

Claim: $u \le v$

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Definitions

If $\lim_{n\to\infty} a_n$ exists, then $u_N \le \lim a_n \le v_N$ so $u \le \lim a_n \le v$.

u and v are useful whether $\lim a_n$ exists or not.

Definition:

$u = \limsup a_n = \lim(\text{lub } \{a_n \mid n > N\})$

and

$v = \liminf a_n = \lim(\text{glb } \{a_n \mid n > N\})$

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lim inf and lim sup

Note: Do not require that $\{a_n\}$ be bounded.

Precautions and Conventions.

1) If $\{a_n\}$ is not bounded above, lub $\{a_n\} = +\infty$ and we define $\limsup a_n = +\infty$

2) If $\{a_n\}$ is not bounded below, glb $\{a_n\} = -\infty$ and we define $\liminf a_n = -\infty$.

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lim inf and lim sup

Is it true that lim sup $\{a_n\}$ =lub $\{a_n\}$?
Not necessarily, because while it is true that

$$\limsup \{a_n\} \leq \text{lub}\{a_n\},$$

some of the values a_n may be much larger than lim sup a_n.

Note that lim sup a_n is the largest value that *infinitely many* a_n's can get close to.

lim inf and lim sup

Theorem: Let $\{a_n\}$ be a sequence of real numbers.

(i) If lim a_n is defined [as a real number, $+\infty$ or $-\infty$, then lim inf a_n =lim a_n = lim sup a_n .

(ii) If lim inf a_n =lim sup a_n, then lim a_n is defined and lim a_n =lim inf a_n =lim sup a_n.

Proof

Let u_N = glb$\{a_n \mid n > N\}$, v_N = lub$\{a_n \mid n > N\}$,

$u = \lim u_N = \liminf a_n$ and

$v = \lim v_N = \limsup a_n$.

(i) Suppose lim $a_n = +\infty$. Let $M > 0$. There is $N \in \mathbf{N}$ so that if $n > N$ then $a_n > M$. Then

$$u_N = \text{glb}\{a_n \mid n > N\} \geq M.$$

So if $m > N$ then $u_m \geq M$.
Therefore lim u_N = lim inf $a_n = +\infty$. Likewise, lim sup $a_n = +\infty$.
Do the case that lim $a_n = -\infty$ similarly.

Proof

Suppose that lim $a_n = L$ ε **R**. Let ε > 0. There is N ε **N** so that $|a_n - L| < ε$ for $n > N$.
$a_n < L + ε$ for $n > N$.
Thus $v_N = \text{lub}\{a_n \mid n > N\} \leq L + ε$.
If $m > N$ then $v_m \leq L + ε$ for all ε > 0.
Thus lim sup $a_n \leq L$ =lim a_n.
Similarly, show that lim $a_n \leq$ lim inf a_n .
Since lim inf $a_n \leq$ lim sup a_n , we have
lim inf a_n = lim a_n = lim sup a_n.

Proof

(ii) If lim inf a_n = lim sup $a_n = \pm\infty$ easy to show that
lim $a_n = \pm\infty$.
Suppose that lim inf a_n = lim sup a_n = L. We need to show that
lim a_n = L.
Let ε > 0. Since L = lim v_N there is an N_0 ε **N** so that
$|L - \text{lub}\{a_n \mid n > N_0\}| < ε$.
Thus, $\text{lub}\{a_n \mid n > N_0\} < L + ε$ and
$a_n < L + ε$ for all $n > N_0$.

Proof

Similarly, since L = lim u_N there is N_1 ε **N** so that
$|L - \text{glb}\{a_n \mid n > N_1\}| < ε$.
Thus, $\text{glb}\{a_n \mid n > N_1\} > L - ε$ and
$a_n > L - ε$ for all $n > N_1$.
These imply $L - ε < a_n < L + ε$ for
$n > \max\{N_0, N_1\}$.
Equivalently, $|a_n - L| < ε$ for $n > \max\{N_0, N_1\}$
This proves that lim a_n = L.

lim inf and lim sup

This tells us that if $\{a_n\}$ converges, then
lim inf a_n = lim sup a_n,
so for large N the numbers lub $\{a_n \mid n > N\}$ and glb$\{a_n \mid n > N\}$ must be close together. This means that all of the numbers in the set $\{a_n \mid n > N\}$ must be close together.

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Theorems

Lemma:
Convergent sequences have the Cauchy property.

Proof:
Suppose that $\lim a_n = L$.

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L|$$

Let $\varepsilon > 0$, there is an integer N so that if $k > N$, $|a_k - L| < \varepsilon/2$. If $m,n > N$ then

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $\{a_n\}$ has the Cauchy property.

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Theorem

Theorem:
A sequence is a convergent sequence if and only if it has the Cauchy property.

Proof: The previous lemma proves half of this.
Show: any sequence with the Cauchy property must converge. Let $\{a_n\}$ have the Cauchy property. We know it is bounded by the previous lemma.
Show: lim inf a_n = lim sup a_n.

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Proof

Let ε > 0. Since $\{a_n\}$ has the Cauchy property, there is an N ϵ **N** so that if $m,n > N$ then $| a_n - a_m| < ε$. In particular, $a_n < a_m + ε$ for all $m,n > N$. This shows that $a_m + ε$ is an upper bound for $\{a_n \mid n > N\}$. Thus

$v_N = \text{lub}\{a_n \mid n > N\} \le a_m + ε$ for $m > N$.

This shows that $v_N - ε$ is a lower bound for $\{a_m \mid m > N\}$, so $v_N - ε \le \text{glb}\{a_m \mid m > N\} = u_N$.

Proof

Therefore

$$\limsup a_n \le v_N \le u_N + ε \le \liminf a_n + ε$$

Since this holds for all ε > 0, we have that

$$\limsup a_n \le \liminf a_n$$

This is enough to give us that the two quantities are equal.

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = \sqrt{a_n + \frac{1}{a_n}}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = 3 - \frac{1}{a_n}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = a_n^{\ 2} + \frac{1}{4}$$