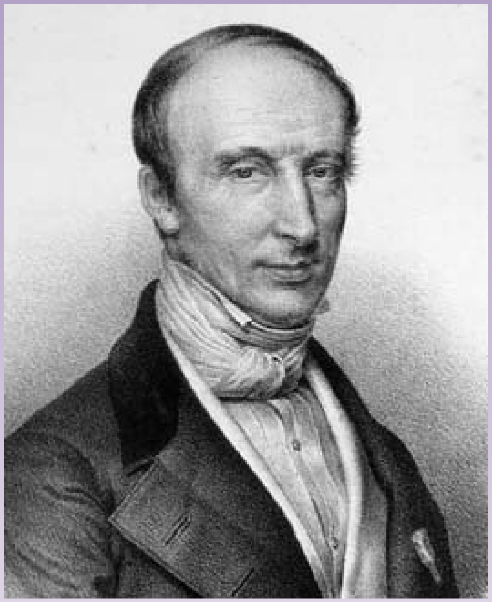
MATH 6101
Fall 2008

The Cauchy Property



$+\infty$ and $-\infty$

1) They are ***not*** real numbers and do ***not*** necessarily obey the rules of arithmetic for real numbers.
2) We often act as if they do.
3) We need guidelines.

Add $+\infty$ and $-\infty$ to **R** and extend the ordering by

$$-\infty < a < +\infty$$

for every real number $a \in \mathbf{R} \cup \{+\infty, -\infty\}$.

$+\infty$ and $-\infty$

If $a \in \mathbf{R}$ then we define the following

1) $a + \infty = +\infty$

2) $a - \infty = -\infty$

3) If $a > 0$, then $a \times \infty = \infty$ and $a \times -\infty = -\infty$

4) If $a < 0$, then $a \times \infty = -\infty$ and $a \times -\infty = +\infty$

We may adopt the following conventions:

$$a/\infty = 0 \text{ and } a/(-\infty) = 0$$

Limits of Sequences

Limit of $\{a_n\}$ exists IFF we can compute L.

Will this always work?

Can we always find the limit?

Do we have to be able to find the limit as a number?

Theorem

Theorem (last lecture): *Every convergent sequence is bounded.*

Is the converse true?

Is it true that every bounded sequence converges?

Find a proof or a counterexample.

Definitions

A sequence $\{a_n\}$ is ***increasing*** if $a_n \le a_{n+1}$ for every n.

A sequence $\{a_n\}$ is ***decreasing*** if $a_n \ge a_{n+1}$ for every n.

A sequence is *monotone* (*monotonic*) if it is either increasing or decreasing.

Examples

1) Find an example of an increasing sequence.

2) Find an example of a decreasing sequence.

3) Find an example of a sequence that is not monotonic.

Increasing Sequences

Decreasing Sequences

Non-monotonic Sequences

Monotone Convergence Theorem

Theorem: *Every bounded monotonic sequence converges.*

Proof:

Let $\{a_n\}$ be a bounded increasing sequence and let $S = \{a_n \mid n \in N\}$. Since the sequence is bounded, $a_n < M$ for some real number M and for all n.

Therefore S is bounded and has a least upper bound. Let $u = \text{lub } S$ and let $\varepsilon > 0$.

Theorem

Proof:

Since u=lub S and $\varepsilon > 0$, $u - \varepsilon$ is ***not*** an upper bound for S. Thus there is an integer K so that $a_K > u - \varepsilon$. Since $\{a_n\}$ is increasing then for all $n > K$, $a_n \geq a_N$ and for all $n > K$

$$u - \varepsilon < a_n \leq u.$$

Thus, $| a_n - u| < \varepsilon$ for all $n > K$ and lim $a_n = u =$ lub S.

Consequences

1) The decimal representation of a real number converges.

$$m < m.d_1d_2d_3d_4\ldots = m + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \leq m + 1$$

Let $a_n = m.d_1d_2d_3d_4\ldots d_n$. Then $a_n \leq a_{n+1}$ so $\{a_n\}$ is increasing.

2) Let $a_0 = 1$ and $a_{n+1} = 1/(1+ a_n)$

Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1 + \sqrt{a_n}$.

Does it converge? Is it monotone?

$a_0 = 1 \quad a_1 = 1 + \sqrt{a_0}) = 2$

$a_2 = 1 + \sqrt{a_1}) = 1 + \sqrt{2} \approx 2.4142\ldots$

$a_3 = 1 + \sqrt{a_2} = 1 + \sqrt{2.4142\ldots} \approx 2.55377\ldots$

Prove it is increasing by induction on *n*.

Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1 + \sqrt{a_n}$.
Converges by Monotone Convergence Theorem. To what does it converge?

Assume: $\lim_{n\to\infty} a_n = L$

$a_{n+1} = 1 + \sqrt{a_n}$

$\lim_{n\to\infty} a_{n+1} = 1 + \lim_{n\to\infty} \sqrt{a_n}$

$L = 1 + \sqrt{(\lim_{n\to\infty} a_n)}$

$L = 1 + \sqrt{L}$

$(L - 1)^2 = L$ so $L^2 - 3L + 1 = 0$

$L = (3 \pm \sqrt{(9 - 4)})/2 = (3 \pm \sqrt{5})/2$

Which one is it? It cannot be both. Why?

Theorem

<u>**Theorem**</u>**:** Let $\{a_n\}$ be a sequence of real numbers.

(i) If $\{a_n\}$ is an unbounded monotonically increasing sequence, then $\lim a_n = +\infty$.

(ii) If $\{a_n\}$ is an unbounded monotonically decreasing sequence, then $\lim a_n = -\infty$.

Theorem

Theorem: Suppose that $\{a_n\}$ is a monotone increasing sequence and $\{b_n\}$ is a monotone decreasing sequence such that

$$a_n \le b_n \text{ for all n = 0,1,2,...}$$

and

$$\{a_n - b_n\} \to 0$$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Theorem

<u>Theorem</u>: *Every sequence contains a monotone subsequence.*

Proof: Let $\{a_n\}$ be a sequence. We say that a term a_n is *dominating* if $a_n > a_m$ for all $m > n$.

Claim: Every sequence contains an infinite number or a finite number of dominating terms. (Note: finite could be 0.)

Theorem

Proof (continued):

(i) Assume $\{a_n\}$ has an infinite number of dominating terms. Call these $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$ where $n_0 < n_1 < n_2 < \ldots$. By definition

$$a_{n_0} > a_{n_1} > a_{n_2} > \ldots$$

which is the monotone subsequence

Theorem

Proof (continued):

(ii) Assume $\{a_n\}$ has a finite number of dominating terms. Thus, there is an m so that for every $n > m$, a_n is not dominating.

That means that for each $n > m$ there exists a $k > n$ so that $a_n \leq a_k$. Let $n_0 = m$. By the above there is a $n_1 > n_0$ so that $a_{n_0} \leq a_{n_1}$. Since $n_1 > n_0$ then there is $n_2 > n_1$ so that $a_{n_1} \leq a_{n_2}$. This gives

$$a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \ldots$$

which is the required monotone subsequence.

Bolzano-Weierstrauss Theorem

Theorem: *Every bounded sequence has a convergent subsequence.*

The Cauchy Property

Definition 1: A sequence $\{a_n\}$ is said to have the Cauchy property if for every $\varepsilon > 0$ there is an index K so that

$$|a_{n+m} - a_n| < \varepsilon$$

for all $n \geq K$ and $m = 1,2,3,\ldots$

[Note: equivalent statement –

$\{a_{n+m}\}^{\infty}_{m=0} \subset (a_n - \varepsilon, a_n + \varepsilon)$ for all $n \geq K$.]

The Cauchy Property

Definition 2: A sequence $\{a_n\}$ is said to have the Cauchy property if for every $\varepsilon > 0$ there is an index K so that if $n,m > K$ then

$$| a_m - a_n| < \varepsilon.$$

Definitions

Let $\{a_n\}$ be bounded – convergent or not, it does not matter.

Limiting behavior of $\{a_n\}$ depends only on the *tails* of the sequence, $\{a_n \mid n > N\}$.

Let $u_N = \text{glb}\{a_n \mid n > N\}$

Let $v_N = \text{lub}\{a_n \mid n > N\}$

FACT: If $\lim a_n$ exists, then it lies in $[u_N, v_N]$.

Definitions

As N increases, the sets $\{a_n \mid n > N\}$ get smaller. Thus,

$$u_1 \leq u_2 \leq u_3 \leq \ldots \text{ and } v_1 \geq v_2 \geq v_3 \geq \ldots$$

Let

$u = \lim_{N\to\infty} u_N$ and $v = \lim_{n\to\infty} v_N$

Both exist – Why?

Claim: $u \leq v$

Definitions

If $\lim_{n\to\infty} a_n$ exists, then $u_N \le \lim a_n \le v_N$
so $u \le \lim a_n \le v$.

u and v are useful whether $\lim a_n$ exists or not.

Definition:

$$u = \limsup a_n = \lim(\text{lub}\,\{a_n \mid n > N\})$$

and

$$v = \liminf a_n = \lim(\text{glb}\,\{a_n \mid n > N\})$$

lim inf and lim sup

Note: Do not require that $\{a_n\}$ be bounded.

Precautions and Conventions.

1) If $\{a_n\}$ is not bounded above, lub $\{a_n\} = +\infty$ and we define lim sup $a_n = +\infty$

2) If $\{a_n\}$ is not bounded below, glb $\{a_n\} = -\infty$ and we define lim inf $a_n = -\infty$.

lim inf and lim sup

Is it true that lim sup $\{a_n\}$ =lub $\{a_n\}$?

Not necessarily, because while it is true that

$$\limsup \{a_n\} \leq \text{lub} \{a_n\},$$

some of the values a_n may be much larger than lim sup a_n.

Note that lim sup a_n is the largest value that *infinitely many* a_n's can get close to.

lim inf and lim sup

<u>Theorem</u>: Let $\{a_n\}$ be a sequence of real numbers.

(i) If $\lim a_n$ is defined [as a real number, $+\infty$ or $-\infty$, then $\liminf a_n = \lim a_n = \limsup a_n$.

(ii) If $\liminf a_n = \limsup a_n$, then $\lim a_n$ is defined and $\lim a_n = \liminf a_n = \limsup a_n$.

Proof

Let $u_N = \text{glb}\{a_n \mid n > N\}$, $v_N = \text{lub}\{a_n \mid n > N\}$,

$$u = \lim u_N = \liminf a_n \text{ and}$$

$$v = \lim v_N = \limsup a_n.$$

(i) Suppose $\lim a_n = +\infty$. Let $M > 0$. There is $N \in \mathbf{N}$ so that if $n > N$ then $a_n > M$. Then

$$u_N = \text{glb}\,\{a_n \mid n > N\} \geq M.$$

So if $m > N$ then $u_m \geq M$.

Therefore $\lim u_N = \liminf a_n = +\infty$. Likewise, $\limsup a_n = +\infty$.

Do the case that $\lim a_n = -\infty$ similarly.

Proof

Suppose that $\lim a_n = L \in \mathbf{R}$. Let $\varepsilon > 0$. There is $N \in \mathbf{N}$ so that $|a_n - L| < \varepsilon$ for $n > N$.

$a_n < L + \varepsilon$ for $n > N$.

Thus $v_N = \text{lub}\{a_n \mid n > N\} \le L + \varepsilon$.

If $m > N$ then $v_m \le L + \varepsilon$ for all $\varepsilon > 0$.

Thus $\limsup a_n \le L = \lim a_n$.

Similarly, show that $\lim a_n \le \liminf a_n$.

Since $\liminf a_n \le \limsup a_n$, we have

$$\liminf a_n = \lim a_n = \limsup a_n.$$

Proof

(ii) If lim inf a_n = lim sup a_n = $\pm\infty$ easy to show that
lim $a_n = \pm\infty$.
Suppose that lim inf a_n = lim sup a_n = L. We need to show that
lim a_n = L.
Let $\varepsilon > 0$. Since $L = \lim v_N$ there is an $N_0 \in \mathbf{N}$ so that

$$|L - \text{lub}\{a_n \mid n > N_0\}| < \varepsilon.$$

Thus, $\text{lub}\{a_n \mid n > N_0\} < L + \varepsilon$ and

$$a_n < L + \varepsilon \text{ for all } n > N_0.$$

Proof

Similarly, since $L = \lim u_N$ there is $N_1 \in \mathbf{N}$ so that

$$|L - \text{glb}\{a_n \mid n > N_1\}| < \varepsilon.$$

Thus, $\text{glb}\{a_n \mid n > N_1\} > L - \varepsilon$ and

$$a_n > L - \varepsilon \text{ for all } n > N_1.$$

These imply $L - \varepsilon < a_n < L + \varepsilon$ for $n > \max\{N_0, N_1\}$.

Equivalently, $| a_n - L| < \varepsilon$ for $n > \max\{N_0, N_1\}$

This proves that $\lim a_n = L$.

lim inf and lim sup

This tells us that if $\{a_n\}$ converges, then

$\liminf a_n = \limsup a_n$,

so for large N the numbers lub $\{a_n \mid n > N\}$ and glb$\{a_n \mid n > N\}$ must be close together. This means that all of the numbers in the set

$\{a_n \mid n > N\}$ must be close together.

Theorems

Lemma:

Convergent sequences have the Cauchy property.

Proof:

Suppose that lim $a_n = L$.

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L|$$

Let ε > 0, there is an integer N so that if $k > N$,

$|a_k - L| < ε/2$. If $m,n > N$ then

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < ε/2 + ε/2 = ε.$$

Thus, $\{a_n\}$ has the Cauchy property.

Theorem

Theorem:

A sequence is a convergent sequence if and only if it has the Cauchy property.

Proof: The previous lemma proves half of this.

Show: any sequence with the Cauchy property must converge. Let $\{a_n\}$ have the Cauchy property. We know it is bounded by the previous lemma.

Show: $\liminf a_n = \limsup a_n$.

Proof

Let ε > 0. Since $\{a_n\}$ has the Cauchy property, there is an $N \in \mathbf{N}$ so that if $m,n > N$ then

$| a_n - a_m| < \varepsilon$. In particular, $a_n < a_m + \varepsilon$ for all $m,n > N$. This shows that $a_m + \varepsilon$ is an upper bound for $\{a_n \mid n > N\}$. Thus

$v_N = \mathrm{lub}\{a_n \mid n > N\} \leq a_m + \varepsilon$ for $m > N$.

This shows that $v_N - \varepsilon$ is a lower bound for

$\{a_m \mid m > N\}$, so $v_N - \varepsilon \leq \mathrm{glb}\{a_m \mid m > N\} = u_N$.

Proof

Therefore

$$\limsup a_n \leq v_N \leq u_N + \varepsilon \leq \liminf a_n + \varepsilon$$

Since this holds for all $\varepsilon > 0$, we have that

$$\limsup a_n \leq \liminf a_n$$

This is enough to give us that the two quantities are equal.

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = \sqrt{a_n + \frac{1}{a_n}}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = 3 - \frac{1}{a_n}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = a_n^{\ 2} + \frac{1}{4}$$