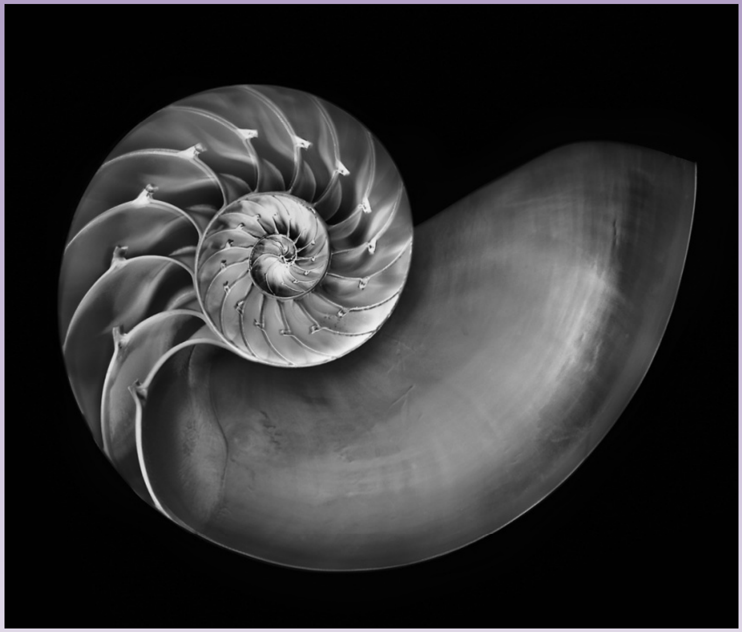
MATH 6101
Fall 2008

Infinite Series and Convergence



Definition

Given any sequence $\{a_n\}$ we associate a new sequence $\{s_n\}$ of *partial sums*:

$$s_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$$

We define the ***series*** $\boldsymbol{\sum a_n}$ to be the limit:

$$\sum a_n = \lim_{n\to\infty} s_n$$

If the sequence of partial sums converges, we say that the infinite series *converges*. Otherwise, we say that the series is *divergent*.

Examples

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2 \qquad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6 \qquad \sum_{n=1}^{\infty} \frac{n^3}{2^n} = 26$$

Examples

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

In this case we have seen that:

$$s_n = \sum_{k=0}^{n} a^k = 1 + a + a^2 + a^3 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

In this case we have seen that:

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n-1)\cdot n}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 - \frac{1}{n}$$

Other Examples

Do these converge or diverge?

$$1+2+3+4+5+\cdots+n+\cdots=\sum_{n=1}^{\infty} n$$

$$1+1+1+1+1+\cdots=\sum_{n=1}^{\infty} 1$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

First Series

$$\sum_{k=1}^{\infty} k = 1+2+3+4+5+\cdots+k+\cdots$$

$$s_n = 1+2+3+4+\cdots+n$$

$$s_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = +\infty$$

Thus, the limit of the sequence of partial sums does not exist as a real number, and the series diverges.

Second Series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \cdots$$

$$s_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n = +\infty$$

Again, the sequence of partial sums does not exist as a real number, and the series diverges.

Third Series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

This one is more difficult to see, but in 1350 Nicole Oresme proves the following:

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots\\ &= 1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)}_{>\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}}+\cdots\\ &> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\end{aligned}$$

So this one does not add up to a finite number.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Does this converge or diverge?

We know that $2n^2 \geq n(n+1)$ so

$$\frac{2}{n(n+1)} \geq \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2$$

Therefore, it does converges.

Continued

We noted earlier that Euler proved in 1735 that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$$

We also know more:

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{2^{2k-1}\left|B_{2k}\right|\pi^{2k}}{(2k)!}$$

where B_n is the nth *Bernoulli* number. Euler only went through the exponent 26.

Bernoulli Numbers

The Bernoulli numbers B_n were discovered by Jakob Bernoulli in conjunction with computing the sums of powers:

$$\sum_{k=0}^{m-1} k^n = 0^n + 1^n + 2^n + 3^n + 4^n + \cdots (m-1)^n$$

For example:

$$\sum_{k=0}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{k=0}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Bernoulli Numbers

$$\sum_{k=0}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$\sum_{k=0}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\sum_{k=0}^{n} k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$\sum_{k=0}^{n} k^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$\sum_{k=0}^{n} k^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$\sum_{k=0}^{n} k^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

Bernoulli Numbers

$$\sum_{k=0}^{n} k^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2$$

$$\sum_{k=0}^{n} k^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

$$\sum_{k=0}^{n} k^{11} = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

$$\sum_{k=0}^{n} k^{12} = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^4 - \frac{691}{2730}n$$

Bernoulli Numbers

Bernoulli then states:

$$\sum_{k=0}^{n} k^p = \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \frac{p}{2}An^{p-1} + \frac{p(p-1)(p-2)}{2\cdot 3\cdot 4}Bn^{p-3}$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{p-5}$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{p-7} + \cdots$$

where

$$A = \frac{1}{6}, \quad B = -\frac{1}{30}, \quad C = \frac{1}{42}, \quad D = -\frac{1}{30}$$

Bernoulli Numbers

The *p*th Bernoulli number is the coefficient of n in the polynomial describing $\sum k^p$.

Other techniques for generating the Bernoulli numbers come from

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k} B_k n^{p+1-k}$$

or

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

Bernoulli Numbers

n	B_n	n	B_n
0	1	12	−691/2730
1	−1/2	14	7/6
2	1/6	16	−3617/510
4	−1/30	18	43867/798
6	1/42	20	−174611/330
8	−1/30	22	854513/138
10	5/66	24	−236364091/2730

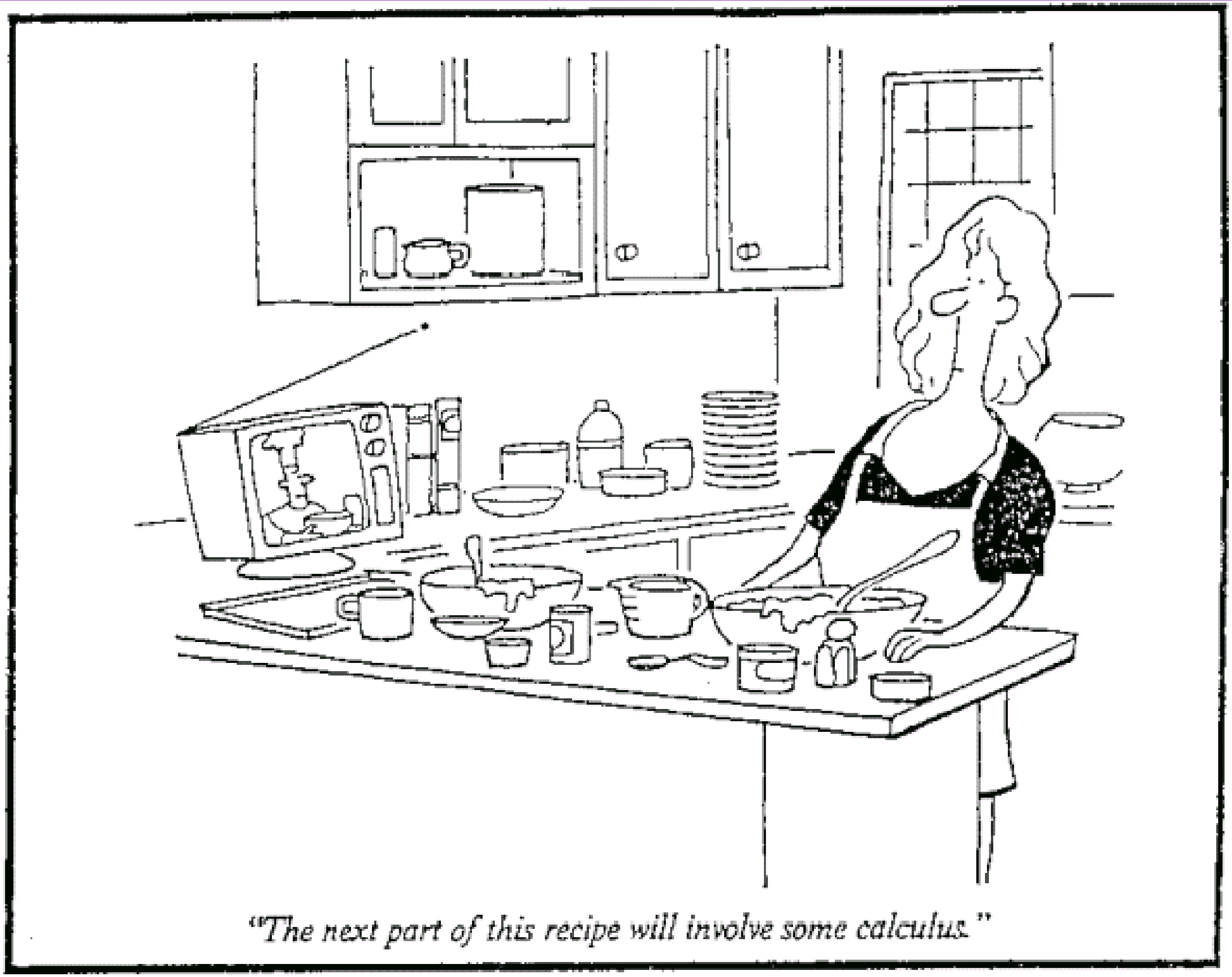
$$B_{2n+1} = 0, \quad n > 0$$

Series

We will be able to show later that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. The easier proof requires a little calculus.



"The next part of this recipe will involve some calculus."

Absolute Convergence

If the terms a_n of an infinite series $\sum a_n$ are all nonnegative, then the partial sums $\{s_n\}$ form a non-decreasing sequence.

Therefore, $\sum a_n$ either converges or diverges to ∞.

$\sum |a_n|$ is non-decreasing for any sequence.

The series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ converges.

Conditional Convergence

A series *converges conditionally*, if it converges, but not absolutely.

- The series $\sum(-1)^n$ diverges.
- The series $\sum(1/2)^n$ converges absolutely.
- The series $\sum (-1)^{n+1}/n$ converges conditionally (this series is called alternating harmonic series).

Order of Summation

Theorem:

(i) *Let* $\sum a_n$ *be an absolutely convergent series. Then any rearrangement of terms in that series results in a new series that is also absolutely convergent to the same limit.*

(ii) *Let* $\sum a_n$ *be a conditionally convergent series. Then, for any real number c there is a rearrangement of the series such that the new resulting series will converge to c.*

(To be proven later)

Algebra of Series

Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. Then

(i) The sum of the two series is again absolutely convergent. $\sum(a_n + b_n) = \sum a_n + \sum b_n$

(ii) The difference of the two series is again absolutely convergent. $\sum(a_n - b_n) = \sum a_n - \sum b_n$

(iii) The product of the two series is again absolutely convergent. Its limit is the product of the limit of the two series.

Algebra of Series

The Cauchy product of two series $\sum a_n$ and $\sum b_n$ is defined as follows. The Cauchy product is

$$\left(\sum_{n=m}^{\infty} a_n\right)\cdot\left(\sum_{n=m}^{\infty} b_n\right)=\left(\sum_{n=m}^{\infty} c_n\right) \text{ where } c_n=\sum_{k=0}^{n} a_k b_{n-k}$$

nth Term Test

Theorem: *If $\sum a_n$ converges then $\{a_n\} \to 0$.*

Metaproof: If $\sum a_n$ converges, then the sequence of partial sums converges $\{s_n\} \to L$. Note that the sequence $\{s_{n-1}\}$ also converges to L. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(s_n - s_{n-1}\right) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

Corollary: *If $|a| \geq 1$ then $\sum a^n$ diverges.*

nth Term Test: *If $\lim a_n \neq 0$, then $\sum a_n$ diverges.*

Comparison Test

Theorem: *If* $\sum a_n$ *and* $\sum b_n$ *are series so that*

$$0 \le a_n \le b_n.$$

Then

if $\sum b_n$ *converges so does* $\sum a_n$;

if $\sum a_n$ *diverges so does* $\sum b_n$.

Comparison Test

Proof: Set

$$d_n = a_0 + a_1 + a_2 + \ldots + a_n$$

$$e_n = b_0 + b_1 + b_2 + \ldots + b_n$$

$\{d_n\}$ and $\{e_n\}$ are increasing sequences.

$$0 \le d_n \le e_n$$

Comparison Test

Each converges or diverges depending on whether it is bounded or not.

$\sum b_n$ converges $\Longrightarrow \{e_n\}$ converges $\Longrightarrow \{e_n\}$ bounded $\Longrightarrow \{d_n\}$ bounded $\Longrightarrow \{d_n\}$ converges $\Longrightarrow \sum a_n$ converges

$\sum a_n$ diverges $\Longrightarrow \{d_n\}$ diverges $\Longrightarrow \{d_n\}$ unbounded $\Longrightarrow \{e_n\}$ unbounded $\Longrightarrow \{e_n\}$ diverges $\Longrightarrow \sum b_n$ diverges

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$ converges or diverges?

$$\frac{1}{n2^n} < \frac{1}{2^n}$$ and we know that the latter converges

Limit Comparison Test

Theorem:

Let $\sum a_n$ *and* $\sum b_n$ *be two series. Suppose also*

$$r = \lim |a_n/b_n| \text{ exists and } 0 < r < +\infty.$$

Then $\sum a_n$ *converges absolutely if and only if* $\sum b_n$ *converges absolutely.*

Limit Comparison Test

Proof: $r = \lim |a_n/b_n|$ and r is a positive real number. There are constants c and C,

$$0 < c < C < +\infty$$

so that for some $N > 1$ if $n > N$

$$c < | a_n/b_n | < C.$$

Assume $\sum a_n$ converges absolutely. For $n > N$, $c|b_n| < |a_n|$. Therefore, $\sum b_n$ converges absolutely by the Comparison Test.

Limit Comparison Test

Assume that $\sum b_n$ converges absolutely.

For $n > N$, $|a_n| < C|b_n|$.

$C \sum b_n$ converges absolutely.

$\sum a_n$ converges absolutely by Comparison Test.

Cauchy Condensation Test

Theorem:

Suppose $\{a_n\}$ is a decreasing sequence of positive terms. Then the series $\sum a_n$ converges if and only if the series $\sum 2^k a_{2^k}$ converges.

p-series Test

Corollary:

For a positive number p, $\sum 1/n^p$ converges if and only if $p > 1$.

p-series Test

Proof:

If $p < 0$ then the sequence $\{1/n^p\}$ diverges to infinity. Hence, the series diverges by the nth Term Test.

If $p > 0$ then consider the series

$$\sum 2^n a_{2^n} = \sum 2^n/(2^n)^p = \sum (2^{1-p})^n.$$

By the geometric series,

- if $0 < p \leq 1$, $2^{1-p} \geq 1$, so right-hand series diverges;
- if $p > 1$ then $2^{1-p} < 1$, so right-hand series converges.

Now the result follows from the Cauchy Condensation Test.

Root Test

Theorem:

Let $\sum a_n$ be a series and let

$$\alpha = \limsup |a_n|^{1/n}.$$

The series $\sum a_n$

i. converges absolutely if $\alpha < 1$,

ii. diverges if $\alpha > 1$.

iii. Otherwise $\alpha = 1$ and the test gives no information.

Root Test

Proof:

i) Suppose that $\alpha < 1$. Then choose an $\varepsilon > 0$ so that $\alpha + \varepsilon < 1$. By definition of lim sup there exists N so that $\alpha - \varepsilon < \text{lub}\{|a_n|^{1/n} \mid n > N, \alpha + \varepsilon\}$. In particular, $|a_n|^{1/n} < \alpha + \varepsilon$ for $n > N$, so $|a_n| < (\alpha + \varepsilon)^n$ for $n > N$. Since $0 < \alpha + \varepsilon < 1$, the geometric series $\sum(\alpha + \varepsilon)^n$ converges. By the Comparison Test $\sum|a_n|$ converges. This means that $\sum a_n$ also converges.

Root Test

Proof:

i) If $\alpha > 1$, then there is a subsequence of $|a_n|^{1/n}$ that has limit $\alpha > 1$. That means that $|a_n| > 1$ for infinitely many n. The sequence $\{a_n\}$ cannot converge to 0, so $\sum a_n$ cannot converge.

ii) For the series $\sum 1/n$ and $\sum 1/n^2$, $\alpha = 1$. The harmonic series diverges and the other converges, so $\alpha = 1$ can not guarantee either convergence or divergence of the series.

Ratio Test

Theorem:

The series $\sum a_n$

i. converges absolutely if $\limsup |a_{n+1}/a_n| < 1$,

ii. diverges if $\liminf |a_{n+1}/a_n| > 1$.

iii. Otherwise
$\liminf |a_{n+1}/a_n| \le 1 \le \limsup |a_{n+1}/a_n|$
and the test gives no information.

Alternating Series Test

Theorem:

If $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$ and $\{a_n\}$ converges to zero, then the alternating series $\sum(-1)^n a_n$ converges.

Problems

$$\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max\{a,b\} \text{ for } a \geq 0 \text{ and } b \geq 0.$$

$$\lim_{n\to\infty} \left(n - \sqrt{n+a}\sqrt{n+b}\right) = -\frac{a+b}{2}$$

$$\lim_{n\to\infty} \frac{a^n - b^n}{a^n + b^n} = \begin{cases} 0 & \text{if } a = b \neq 0 \\ 1 & \text{if } |a| > |b| \\ -1 & \text{if } |a| < |b| \\ \text{undefined} & \text{if } a = b = 0 \end{cases}$$

Problems

$$\lim_{n\to\infty}\frac{2^{n^2}}{n!}$$

$$2n^2 \geq n(n+1) \Rightarrow n^2 \geq \frac{n(n+1)}{2} = n + (n-1) + \cdots + 2 + 1$$

$$2^{n^2} \geq 2^{\frac{n(n+1)}{2}} = 2^n \times 2^{n-1} \times \cdots \times 2^2 \times 2$$

$$\frac{2^{n^2}}{n!} \geq \frac{2^n \times 2^{n-1} \times \cdots \times 2^2 \times 2}{n \times (n-1) \times \cdots \times 2 \times 1} \to \infty$$

Problems

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$$

Since $\sin n\theta \leq 1$ for all n we have that

$$\frac{\sin(n\theta)}{n^2} \leq \frac{1}{n^2}$$

so this series converges by the Comparison Test with the series .

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Problems

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$$

This diverges by the Comparison Test with the series

$$\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}$$

which diverges by the *p*-series Test.

Problems

$$\sum_{n=1}^{\infty}(-1)^n \frac{\log n}{n}$$

Since lim $(\log n)/n = 0$ and $a_{n+1} \leq a_n$, by the Alternating Series Test this series converges.

Problems

$\sum_{n=1}^{\infty} \frac{n^2}{n!}$ Use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2} = \lim_{n\to\infty}\frac{n+1}{n^2} = 0$$

This series converges.

Problems

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

$$\frac{\log n}{n} > \frac{1}{n}$$

So the series diverges by the Comparison Test.

Problems

$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

$$\log n < n \Rightarrow \frac{1}{n} < \frac{1}{\log n}$$

So the series diverges by the Comparison Test.

Problems

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$$

Use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

So the series converges by the Root Test.

Problems

$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$$

This series diverges by the Limit Comparison Test with $\sum 1/n$.

Problems

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$$

$$\frac{1}{n^2 \log n} < \frac{1}{n^2}$$

So the series converges by the Comparison Test.

Problems

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

Use the Limit Comparison Test with $\sum 1/n$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{n}{n \cdot n^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1 < +\infty$$

Thus, the series diverges.