Topology
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Chapter 1

Introduction to Topology

1.1 History[1]

Topology is thought of as a discipline that has emerged in the twentieth century. There are precursors of topology dating back into the 1600's. Gottfried Wilhelm Leibniz (1646–1716) was the first to foresee a geometry in which position, rather than magnitude was the most important factor. In 1676 Leibniz use the term *geometria situs*(geometry of position) in predicting the development of a type of vector calculus somewhat similar to topology as we see it today.

The first practical application of topology was made in the year 1736 by the Swiss mathematician Leonhard Euler (1707–1783) in the Königsberg Bridge Problem.

Carl F. Gauss (1777–1855) predicted in 1833 that *geometry of location* would become a mathematical discipline of great importance. His studey of closed surfaces such as the sphere and the torus and surfaces much like those encountered in multi-dimensional calculus may be considered as a harbinger of general topology. Gauss was also interested in knots, which are of current interest today in topology.

The word ***topology*** was first used by the German mathematics Joseph B. Listing (1808–1882) in the title of his book *Vorstudien zur Topologie* (*Introductory Studies in Topology*), a textbook published in 1847. Listing book dealt with knots and surfaces but failed to generate much interest in either the name or the subject matter. Throughout much of the nineteenth and early twentieth centuries, much of what now falls under the auspices of topology was studied under the name of *analysis situs* (analysis of position).

Bernard Riemann (1826–1866) was the first mathematician to foresee topology in the generality it has achieved today. He initiated the study of connectivity of a surface, or the arrangement of holes in a surface. He used concepts in which the number of dimensions exceeded three, which at that time was generally conceded to be the maximum number of dimensions involved with any geometric object.

Present-day topology can be traced to two primary sources: the development of non-euclidean geometry and the process of putting calculus on a firm mathematical foundation.

[1]The information here is taken from *Principles of Topology* by Fred H. Croom.

1.2 Sets and Set Operations

We need some basic information about sets in order to study the logic and the axiomatic method. This is not a formal study of sets, but consists only of basic definitions and notation.

Braces { and } are used to name or enumerate sets. The *roster* method for naming sets is simply to list all of the elements of a set between a pair of braces. For example the set of integers 1, 2, 3, and 4 could be named

$$\{1, 2, 3, 4\}.$$

This does not work well for sets containing a large number of elements, though it can be used. The more common method for this is known as the *set builder notation*. A property is specified which is held by all objects in a set. $P(x)$, read *P of x*, will denote a sentence referring to the variable x. For example,

$x = 23$

x is an odd integer.

$1 \leq x \leq 4$.

The set of all objects x such that x satisfies $P(x)$ is denoted by

$$\{x \mid P(x)\}.$$

The set $\{1, 2, 3, 4\}$ can be named

$$\{x \mid 1 \leq x \leq 4,\ x \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid 1 \leq x \leq 4\}.$$

From hence forth, the words *object, element*, and *member* mean the same thing when referring to sets. Sets will be denoted mainly by capital Roman letters and elements of the sets by small letters. The following have the same meaning:

$a \in A$

a is in set A

a is a member of set A

a is an element of set A

Likewise, $a \notin A$ means that a is **not** an element of set A.

A is a *subset* of B if every element of A is also an element of B. The following have the same meaning:

$A \subset B$

Every element of A is an element of B

If $a \in A$, then $a \in B$

A is included in B

B contains A

A is a subset of B

Note that a set is always a subset of itself.

If A and B are sets, then we say that $A = B$ if A and B represent the same set:

$A = B$

A and B are the same set

A and B have the same members

$A \subset B$ and $B \subset A$

The set which contains no elements is known as the *empty set*, and is denoted by $\emptyset$. Note that for each set A, $\emptyset \subset A$.

The *intersection* of two sets A and B is the set of all elements common to both sets. The intersection is symbolized by $A \cap B$ or $\{x \mid x \in A \text{ and } x \in B\}$. The *union* of two sets A and B is the set of elements which are in A or B or both. The union is symbolized by $A \cup B$ or $\{x \mid x \in A \text{ or } x \in B\}$.

1.2.1 Universal Sets and Compliments

When we are working in an area or on a certain problem, we always have a frame of reference in which we are working called a *universal set*. In our geometry course, it will be the set of points that lie on a plane. In calculus we consider the set of real numbers, the set of real functions, the set of differentiable functions, and the set of continuous functions as universal sets.

The *complement* of a set A is defined to be the set of all elements of the universal set which are not in A, and is symbolized by $CA = A' = A^c$. Note that $A \cup A^c$ is always the universal set, while $A \cap A^c = \emptyset$.

The *set difference* of the sets A and B is defined to be all of those elements in A which are not in B. It is denoted by

$$A \setminus B = \{x \in A \mid x \ni B\}.$$

Note that $A \setminus B$ and $B \setminus A$ will usually be different, and that even though $A \setminus B = \emptyset$ it need not follow that $A = B$.

1.3 Products Sets

Let X and Y be sets. The set of all ordered pairs $\{(x, y) \mid x \in X \textit{ and } y \in Y\}$ is the *product set* $X \times Y$, or *Cartesian product* or *direct product* of X and Y. A *slice* of this product set is $\{x\} \times Y$ or $X \times \{y\}$ for a given $x \in X$ or $y \in Y$. Examples of common product spaces are the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, 3-space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, a right circular cylinder, $S^1 \times [0, 1]$, or the torus, $S^1 \times S^1$.

Theorem 1 *Let X and Y be sets and let $A, C \subset X$ and $B, D \subset Y$.*

a) $A \times (B \cap D) = (A \cap B) \times (A \cap D)$.

b) $A \times (B \cup D) = (A \cup B) \times (A \cup D)$.

c) $A \times (Y \setminus D) = (A \times Y) \setminus (A \times D)$.

d) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

e) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

f) $(X \times Y) \setminus (A \times B) = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$

The concept of the product of two sets can be extended to more than two factors. If $\{X_i\}_{i=1}^n$ is a finite collection of sets, then their product is

$$X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \text{ for each } i = 1, 2, \ldots, n\}.$$

For an infinite collection of sets, the product is defined by

$$\prod_{i=1}^{\infty} X_i = \{(x_1, x_2, x_3, \ldots) \mid x_i \in X_i \text{ for each } i = 1, 2, \ldots\}.$$

1.4 Functions

A *function* $f \colon X \to Y$ is a rule which assigns to each $x \in X$ a unique $y \in Y$ and we say $y = f(x)$. If $y = f(x)$ then y is called the *image* of x and x is called the *preimage* of y. The set X is the *domain* of f and Y is the *range* or *codomain* of f.

Let $A \subset X$. The set $f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}$ is called the *image* of A. The set $f(X)$ is called the *image of* f. For $B \subset Y$, the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is the *inverse image* of B under f. The set of points

$$\Gamma = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is called the *graph* of the function f.

A function $f\colon X \to Y$ is *injective* if for distinct elements $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$ in Y. Another way to think of this is to say that f is injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

If $f(X) = Y$, the function f is said to be *surjective*.

A function that is surjective and injective is called a *bijection*. In this case we have that $f\colon X \to Y$ is a bijection provided that each member of Y is the image under f of exactly one member of X. In this case the *inverse function* $f^{-1}\colon Y \to X$ exists assigning to each element $y \in Y$ its unique preimage $x = f^{-1}(y)$ in X.

The *identity function* $i_X\colon X \to X$ from a set X to itself is the function defined by $i_X(x) = x$ for all $x \in X$. This function is often denoted by 1_X.

If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, then the *composite function* $g \circ f\colon X \to Z$ is defined by $g \circ f(x) = g(f(x))$, for $x \in X$.

Definition 1 *Let X be a set. A* ***sequence*** *in X is a function $f\colon \mathbb{Z}^+ \to X$ whose domain is the set of all positive integers, $\mathbb{Z}^+$ or the set of positive integers less than or equal to some given positive integer N. The sequence is called finite if its domain is $\{1, 2, \ldots, N\}$ and infinite if its domain is all positive integers.*

1.5 Equivalence Relations

Let X be a set. A *relation* R on X is a subset of $X \times X$. If $(x, y) \in R$ we will say that x is related to y by R and to write xRy.

A relation R on a set X is called *reflexive*, *symmetric*, or *transitive* if it satisfies the corresponding property below.

(a) *The Reflexive Property*: xRx for all $x \in X$.

(b) *The Symmetric Property*: If xRy, then yRx.

(c) *The Transitive Property*: If xRy and yRz, then xRz.

Definition 2 *An* ***equivalence relation*** *on a set X is a relation on X which is reflexive, symmetric, and transitive.*

Definition 3 *Let $\sim$ denote an equivalence relation on X. For $x \in X$ the set $[x] = \{y \in X \mid y \sim x\}$ is calle dth e equivalence class of x.*

Proposition 1 *Let X be a set and let $\sim$ denote an equivalence relation on X.*

a) $x \in [x]$ for each $x \in X$.

b) $x \sim y$ if and only if $[x] = [y]$.

c) $x \not\sim y$ if and only if $[x] \cap [y] = \emptyset$.

d) For $x, y \in X$, $[x]$ and $[y]$ are either identical or disjoint.

1.6 Cardinality

We are often interested in how big sets are in relation to one another. Clearly, we can tell the difference in sizes of two finite sets, but how do we differentiate between two infinite sets? Are there different sizes of infinite sets? How do we compare sets to tell if one has a greater number of members?

Definition 4 *a set A is* ***finite*** *if A is empty or if there is a bijection between A and the set of integers from 1 to N for some positive integer N. In the latter case, A is said to have N members. If a set is not finite, it is called* ***infinite****.*

Definition 5 *A set is* ***denumerable*** *or* ***countably infinite*** *if there is a bijection between the set and the positive integers. A set which is either finite or denumerable is called* ***countable****. A set which is not countable is called* ***uncountable****.*

Lemma 1 *a) Each subset of a finite set is finite.*

b) Each subset of a countable set is countable.

c) Each set which contains an infinite set is infinite.

d) Each set which contains an uncountable set is uncountable.

Example 1 1. The set $\mathbb{Z}^+ \cup \{0\}$ of all non-negative integers is countable. The bijection is $f\colon \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+$ given by

$$f(n) = n + 1, n \in \mathbb{Z}^+ \cup \{0\}.$$

2. The set of all integers, $\mathbb{Z}$ is countable. The bijection $g\colon \mathbb{Z} \to \mathbb{Z}^+$ is given by

$$g(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is positive} \\ -2n & \text{if } n \text{ is negative} \end{cases}$$

3. The product set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. One method is to use the Cantor Diagonalization Method to count the ordered pairs (m, n). A second method is to define the function $g\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$g(m, n) = 2^m 3^n,\ (m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Now, g is not surjective, but the Fundamental Theorem of Arithmetic on the unique factorization into primes guarantees that the function g is injective. Thus, there is a bijection from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to a subset of $\mathbb{Z}^+$. Since every subset of a countable set is countable, we have that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Theorem 2 *a) If $\{A_i\}_{i=1}^N$ is a finite collection of finite sets, then both $\bigcup_{i=1}^N A_i$ and $\prod_{i=1}^N A_i$ are finite.*
(Finite unions and finite products of finite sets are finite.)

b) If $\{A_i\}_{i=1}^\infty$ is a countable collection of countable sets, then $\bigcup_{i=1}^\infty A_i$ is countable.
(Countable unions of countable sets are countable.)

c) If $\{A_i\}_{i=1}^N$ is a finite collection of countable sets, then both $\prod_{i=1}^N A_i$ is countable.
(Finite products of countable sets are countable.)

Why didn't we claim that a countable product of countable sets is countable? Mainly because it is not true, as is seen in the following example.

Example 2 Let $A_i = \{0, 1\}$ for $i = 1, 2, \dots$. Let

$$U = \prod_{i=1}^{\infty} A_i = \{(a_1, a_2, a_3, \dots) \mid a_i = 0 \text{ or } 1\}.$$

Assume that U is countable. Then there is a bijection $f\colon \mathbb{Z}^+ \to U$. For an element $a = (a_1, a_2, a_3, \dots) \in U$ we shall refer to a_1 as the first coordinate, a_2 as the second coordinate, and so forth.

We can list all of the elements in U using the bijection f. They are $\{f(1), f(2), F(3), \dots\}$. Consider the following element in U. Define $x = (x_1, x_2, x_3, \dots)$ as follows:

$$x_i = \begin{cases} 0 & \text{if the } i^{th} \text{ coordinate of } f(i) \text{ is } 1 \\ 1 & \text{if the } i^{th} \text{ coordinate of } f(i) \text{ is } 0 \end{cases}$$

Then, we have that for each positive integer i, $x \neq f(i)$ for they differ in the i^{th} coordinate. This means that x is not in the exhaustive list of elements we have in our bijection. That is this bijection is not surjective. This contradiction show us that U cannot be countable.

Theorem 3 *The set of rational numbers is countable.*

Proof: There are several ways of proving this. One method is to use the Cantor Diagonalization Method to count the rationals.

(Method 1) List the positive rational numbers in rows where the first row consists of all positive rational numbers with 1 as a denominator, the second row lists all positive rational numbers with 2 as a denominator, and so on:

$$\begin{array}{ccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Clearly, we have each positive rational number in here numerous times, but the diagonalization method will still show that there are a countable number of elements in this array. The positive rationals form a subset of this array, thus there must be a countable number of positive rationals. This will yield that there are a countable number of rational numbers.

(METHOD 2) Every rational number can be expressed uniquely in lowest terms as m/n where m and n are integers with no common positive divisor other than 1, and n is positive. Consider the function $m/n \mapsto (m, n)$ from the set of rational numbers into $\mathbb{Z} \times \mathbb{Z}$. This function is injective since the ordered pair (m, n) determines only one rational number m/n. Thus, the set of rational numbers is equivalent to a subset of the countable set $\mathbb{Z} \times \mathbb{Z}$, and is hence countable. ∎

Theorem 4 *The set of real numbers is uncountable.*

PROOF: We will make use of the example of the countable product above. Each element in $\prod_{i=1}^{\infty} 0, 1$ is a sequence consisting of 0's and 1's. Each of these sequences represents a unique real number between 0 and 1, by the correspondence

$$(a_1, a_2, a_3, \dots) \mapsto 0.a_1a_2a_3\dots.$$

This is a one-to=one correspondence. Thus, the set of real numbers between 0 and 1 representable as a decimal using only 0's and 1's is an uncountable set. Thus, $\mathbb{R}$ contains an uncountable set and hence is uncountable. ∎

Theorem 5 *The set of irrational numbers is uncountable.*

PROOF: Since the set of real numbers is the union of the set of rational numbers and the set of irrational numbers, if the set of irrationals were countable, then we would have that the real numbers are countable. That failing to be true, implies that the irrationals must be uncountable. ∎

Chapter 2

Metric Spaces

2.1 Definition and Some Examples

Definition 6 *Let X be a set and $d\colon X \times X \to \mathbb{R}^+$ a function satisfying the following properties. For all $x, y, z \in X$,*

a) $d(x, y) = 0$ *if and only if* $x = y$.

b) $d(x, y) = d(y, x)$.

c) $d(x, z) \leq d(x, y) + d(y, z)$.

*Then d is called a **metric** or **distance function** on X and $d(x, y)$ is called the **distance** from x to y. The set X with a metric d is called a **metric space** and is denoted by (X, d).*

Note that these properties are modeled on the distance functions that we have on $\mathbb{R}$ and $\mathbb{R}^2$. Doing so we usually call property (c) the *Triangle Inequality.*

Example 3 The real line, $\mathbb{R}$ is a metric space using the standard distance function, the absolute value: $d(a, b) = |a - b|$. The above properties are standard proofs about the absolute value function.

Example 4 The plane, $\mathbb{R}^2$, with the usual Euclidean distance formula is a metric space. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 5 These are special cases of the general Euclidean n-space,

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}.$$

The distance formula here is the usual distance formula for Euclidean n-space:

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}.$$

d is called the *usual metric* on $\mathbb{R}^n$.

To show that d is a metric, we need two standard results about vectors in $\mathbb{R}^n$. First, let $a \in \mathbb{R}^n$. The *norm* $\|a\|$ is the distance from a to the origin $O = (0, 0, \ldots, 0)$:

$$\|a\| = d(a, O) = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}.$$

Theorem 6 (Cauchy-Schwarz Inequality) *For any points $a, b \in \mathbb{R}^n$*

$$|a \cdot b| \leq \|a\|\|b\|.$$

Theorem 7 (The Minkowski Inequality) *For any points $a, b \in \mathbb{R}^n$*

$$\|a + b\| \leq \|a\| + \|b\|.$$

The distance between two points is given by $d(a, b) = \|a - b\|$.

The first two conditions making d a metric are easily seen to be satisfied. We only need check the Triangle Inequality. Let $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x, z) &= \|x - z\| = \|x - y + y - z\| \\ &\leq \|x - y\| + \|y - z\| \\ &= d(x, y) + d(y, z) \end{aligned}$$

Example 6 [The Taxicab Metric] Define a function $d' \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$d'(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is called the taxicab metric because the distance is measured along line segments parallel to the coordinate axes.

Clearly, $d'(x, x) = 0$ and if $d'(x, y) = 0$, then $|x_1 - y_1| + |x_2 - y_2| = 0$ which means $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. This implies that $x_1 = y_1$ and $x_2 = y_2$, and $x = y$. Because of the basic properties of the absolute value, it is obvious that $d'(x, y) = d'(y, x)$. The Triangle Inequality follows because of the validity of the Triangle Inequality with the absolute value on the real line.

What is the following set?

$$U = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid d'(x, O) = 1\}.$$

We can define an analogous metric, called the taxicab metric, on $\mathbb{R}^n$.

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

Example 7 [The Max Metric on $\mathbb{R}^n$] Another metric for $\mathbb{R}^n$ is given by taking the largest of the differences of the coordinates of x and y.

$$d''(x,y) = \max\{|x_i - y_i\}_{i=1}^n.$$

Example 8 [The Discrete Metric] For any set X, define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

This defines a metric on X, called the *discrete metric*. It is usually of little use, except for counterexamples. It does show, though, that every set can be assigned a metric.

Example 9 Let $\mathscr{C}[a,b]$ denote the set of all continuous real-valued functions defined on the interval $[a,b]$. For $f, g \in \mathscr{C}[a,b]$ define

$$\rho(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

The fact that ρ is a metric follows from the usual properties of the Riemann integral. This metric measures the distance between two functions to be the area between the two graphs from $x = a$ to $x = b$.

Example 10 For the set $\mathscr{C}[a,b]$ define ρ' by

$$\rho'(f,g) = \operatorname{lub}\{|f(x) - g(x)| \mid x \in [a,b]\}.$$

The metric is called the *supremum metric* or the *uniform metric* for $\mathscr{C}[a,b]$. It measures the distance between f and g to be the supremum of the vertical distances from points $(x, f(x))$ to $(x, g(x))$ on the graphs of f and g on the closed interval $[a,b]$.

Definition 7 *A number u is an* ***upper bound*** *for a set A of real numbers provided that $a \leq u$ for all $a \in A$. If there is a smallest upper bound u_0 for A, that is an upper bound that is less than or equal to all other upper bounds for A, then u_0 is called the* ***least upper bound or supremum*** *of A. The least upper bound for a set A is denoted by* $\operatorname{lub} A$ *or* $\sup A$.

Definition 8 *A number ℓ is an* ***lower bound*** *for a set A of real numbers provided that $\ell \leq a$ for all $a \in A$. If there is a largest lower bound ℓ_0 for A, that is a lower bound that is less than or equal to all other lower bounds for A, then ℓ_0 is called the* ***greatest lower bound or infimum*** *of A. The greatest lower bound for a set A is denoted by* $\operatorname{glb} A$ *or* $\inf A$.

A very basic property of the real numbers is included in the following two statements:

The Least Upper Bound Property: Every non-empty set of real numbers which has an upper bound has a least upper bound.

The Greatest Lower Bound Property: Every non-empty set of real numbers which has a lower bound has a greatest lower bound.

We will accept the first property as an axiom of the real number system. The second property follows from the first.

Definition 9 *Let (X, d) be a metric space and let A be a non-empty subset of X. If $\{d(x, y) \| x, y \in A\}$ has an upper bound, then A is said to be* ***bounded****, and* $\text{lub}\{d(x, y) \| x, y \in A\}$ *is called the* ***diameter*** *of A. For completeness, we define the diameter of the empty set to be 0. If the set X is bounded, then we call (X, d) a bounded metric space.*

If $x \in X$, then the ***distance*** *from x to A is defined by*

$$d(x, A) = \text{glb}\{d(x, y) \mid y \in A\}.$$

Theorem 8 *Let $\{(X_i, d_i)\}_{i=1}^{n}$ be a finite collection of metric spaces and let*

$$X = \prod_{i=1}^{n} X_i.$$

For each pair of points $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in X, let $d\colon X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (d_i(x_i, y_i))^2 \right)^{1/2}.$$

Then (X, d) is a metric space. The metric d defined above is called the ***product metric*** *on X.*

2.2 Continuous Functions

In topology we are concerned with how spaces are changed when stretched, bent, twisted and modified — but not torn. We do so by studying the maps that do so. Our friend here is the continuous map. In your study of calculus, you saw that continuous functions did many things. At the time you were more interested in special continuous functions — the differential functions. We here are more interested in the more general function.

In calculus, we saw that a continuous function was one that did not do too much damage to the domain in the range. By this, we mean that if two points were close in the domain, then their images were not too far apart in the image. We saw this intuitively through looking at graphs and looking at limits. To insure specificity, we need the definition of continuity due to Cauchy and Weierstrauss. It is one with which you are familiar.

Definition 10 *Let $f\colon (X,d) \to (Y,d')$ be a function between two metric spaces. Let $a \in X$. We say that f is continuous at a if given any $\epsilon > 0$ there is a $\delta > 0$ so that $d'(f(x), f(a)) < \epsilon$ whenever $d(x,a) < \delta$. We say that f is continuous if it is continuous at $a \in X$ for all $a \in X$.*

This clearly depends on the metric in each of the two spaces. A change of metric *might* change the continuity of the given function. Will it? Is continuity that dependent on the metric in the domain or the range?

Let's check two well-known functions that we think should be continuous and make certain that they are continuous under this definition.

Example 11 Let $f\colon (X,d) \to (Y,d')$ be given by $f(x) = b$ for all $x \in X$ where $b \in Y$ is a constant. This is just the *constant function*.

To show that f is continuous, we need to show that if we are given any $\epsilon > 0$, then we can find a $\delta > 0$ so that whenever $d(x_1, x_2) < \delta$ then $d'(f(x_1), f(x_2)) < \epsilon$. In this case, this is easy. This is because $d'(f(x_1), f)(x_2)) = d'(b,b) = 0 < \epsilon$ for any choice of $x_1, x_2 \in X$. Thus, it does not matter what we may choose for δ. You could take $\delta = \epsilon$ or $\delta = 1$. Regardless, whenever $d(x_1, x_2) < \delta$ then $d'(f(x_1), f(x_2)) = 0 < \epsilon$, and we are done.

Example 12 Let $1_X\colon (X,d) \to (X,d)$ denote the identity map from X to itself given by $1_X(x) = x$. We claim that this function is continuous.

Again, to show this we are given an $\epsilon > 0$. We then need to find a $\delta > 0$ so that whenever $d(x_1, x_2) < \delta$ then $d(1_X(x_1), 1_X(x_2)) < \epsilon$. However, since $1_X(x_1) = x_1$ and $1_X(x_2) = x_2$, it is easy to see that if we take $\delta \le \epsilon$, then if $d(x_1, x_2) < \delta$ it follows that $d(1_X(x_1), 1_X(x_2)) = d(x_1, x_2) < \delta \le \epsilon$. Thus, 1_X is a continuous function.

Example 13 This time we will be working with the same underlying set, but we will place a different metric on it. Will this make a difference?

Let $X = \mathbb{R}^n$ with the usual metric. Let $Y = \mathbb{R}^n$ with the maximum metric,

$$d''((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max_{1 \le i \le n} \{|x_i - y_i|\}.$$

Define $h\colon (X,d) \to (Y,d'')$ by $h(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$. It is the identity map on the underlying set, but it does not carry the same metric information. Is h continuous? Is h^{-1} continuous?

It turns out that both are continuous! To prove this, let's first look at $h^{-1}\colon (Y,d'') \to (X,d)$. We are given an $\epsilon > 0$. We need to find a $\delta > 0$ so that if

$$d''((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \delta$$

then $d(h^{-1}(x_1, x_2, \ldots, x_n), h^{-1}(y_1, y_2, \ldots, y_n)) < \epsilon$.

To say that $d''((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \delta$ means that $|x_i - y_i| < \delta$ for all $i = 1, \ldots, n$. Thus,

$$d(h^{-1}(x_1, x_2, \ldots, x_n), h^{-1}(y_1, y_2, \ldots, y_n)) = d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \quad (2.1)$$

$$= \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \quad (2.2)$$

$$< \left(\sum_{i=1}^{n} \delta^2 \right)^{1/2} = \delta\sqrt{n} \quad (2.3)$$

Thus, we need $\delta\sqrt{n} < \epsilon$, or take $\delta < \dfrac{\epsilon}{\sqrt{n}}$.

Now, to show that h is continuous we are given an $\epsilon > 0$. We need to find δ so that whenever $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \delta$, we have that

$$d''((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \epsilon.$$

To say that $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \delta$ means that $\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} < \delta$. Thus, each of the differences $|x_i - y_i|$ must be less than δ, and the largest of these differences is still less than δ. Thus, in order for $d''((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) < \epsilon$, we need only choose $\delta < \epsilon$.

2.3 Open Sets and Closed Sets

Definition 11 *Let (X, d) be a metric space, $a \in X$, and $r > 0$ a positive real number. The **open ball** $B_d(a; r)$ with **center** a and **radius** r is the set*

$$B_d(a; r) = \{x \in X \mid d(a, x) < r\}.$$

When there is only one metric under consideration, we will simplify the notation to $B(a; r)$.

Definition 12 *Let (X, d) be a metric space and let $a \in X$. A subset $N \subset X$ is a **neighborhood of** a if there is a $\delta > 0$ so that $B(a; \delta) \subset N$. The collection $\mathfrak{N}_a$ of all neighborhoods of a point $a \in X$ is called a **complete system of neighborhoods** of the point a.*

Definition 13 *A subset U of a metric space (X, d) is an **open set** with respect to the metric d provided that U is a union of open balls. The family of all open sets defined in this way is called the **topology for** X **generated by** d. A subset $C \subset X$ is said to be **closed** (with respect to d) if its complement $X \setminus C$ is an open set (with respect to d).*

Thus, a neighborhood of a and an open set containing a need not be the same thing. However, if U is an open set containing a, then U is a neighborhood of a.

Theorem 9 *The following statements are equivalent (TFAE) for a subset U of a metric space (X, d).*

a) U is an open set;

b) for each $x \in U$ there is an $\epsilon_x > 0$ so that $B(x; \epsilon_x) \subset U$.

c) for each $x \in U$, $d(x, X \setminus U) > 0$, if $U \neq X$.

Proof: What this means is that Statements (a) and (b) are equivalent, (b) and (c) are equivalent, and (a) and (c) are equivalent. We can show this by proving that (a) is equivalent to (b) and then that (b) is equivalent to (c). In condition (c) we will assume that $U \neq X$ since the distance from the empty set is not defined.

Assume that U is an open set and let $x \in U$. Since U is the union of open balls, then $x \in B(a; r) \subset U$. Then $d(x, a) < r$. We want to center an open ball at x and have it contained in U. Choose $\epsilon_x \leq r - d(x, a)$. Then $B(x; \epsilon_x) \subset B(a; r)$ for the following reason: If $y \in B(x; \epsilon_x)$,

$$d(y, a) \leq d(y, x) + d(x, a) < \epsilon_x + d(x, a) \leq r - d(x, a) + d(x, a) = r.$$

Thus, $B(x; \epsilon_x)$ is an open ball of positive radius centered at x and contained in U. Thus $(a) \implies (b)$.

To show that $(b) \implies (a)$, since each $x \in U$ lies in an open ball contained in U, U is the union of these open balls.

To see that $(b) \implies (c)$, let $B(x; \epsilon_x) \subset U$. Then any point within distance ϵ_x of x is in U, so the distance from x to $X \setminus U$ must be at least ϵ_x. Thus, $d(x, X \setminus U) > 0$ for each $x \in U$.

Assuming that (c) holds, $d(x, X \setminus U) = \alpha_x > 0$ depending on x. This means that the distance from x to a point outside U must be at least α_x, so any point within distance α_x of x must be in U. This means $B(x; \alpha_x) \subset U$. ∎

Note that we have just shown that for each $a \in X$ and for each $\delta > 0$, the open ball $B(a; \delta)$ is a neighborhood of each of its points.

2.3.1 Neighborhoods and Continuous Functions

How do we plan to use this information? While our definition of continuity is precise, it requires some specificity and does not look generalizible. What I mean by this is that the definition seems to rely specifically on the definition of the metric, and it will be hard to realign our definition when we have to move away from metric spaces.

Theorem 10 *Le $f\colon (X, d) \to (Y, d')$. f is continuous at $a \in X$ if and only if for each neighborhood M of $f(a)$ there is a corresponding neighborhood N of a, such that*

$$f(N) \subset M,$$

or equivalently

$$N \subset f^{-1}(M).$$

PROOF: First, let's suppose that f is continuous at $a \in X$ and let M be a neighborhood of $f(a)$. This means that for some $\epsilon > 0$ $B_{d'}(f(a);\epsilon) \subset M$. Since f is continuous at $a \in X$ we know that we can find $\delta > 0$ so that if $d(x,a) < \delta$ then $d'(f(a), f(x)) < \epsilon$. This says that $B_d(a;\delta) \subset f^{-1}(M)$ and we already have seen that $B_d(a;\delta)$ is a neighborhood of $a \in X$. Thus, if f is continuous, we have found a corresponding neighborhood to M.

Now, suppose that for any neighborhood, M, of $f(a)$ we can find a neighborhood N of a so that $f(N) \subset M$. Let $\epsilon > 0$ be given to you. You must find a $\delta > 0$ so that whenever $d(a,x) < \delta$ we have $d'(f(a), f(x)), \epsilon$. Now, let $M = B_{d'}(f(a);\epsilon)$. M is a neighborhood of $f(a)$, so we know that there is a neighborhood $N \subset X$ of a so that $f(N) \subset M$. Since N is a neighborhood of a, it must contain a δ-ball centered at a, $B_d(a;\delta) \subset N$, by the definition of a neighborhood. Thus, if we have $x \in B_d(a;\delta)$, then $f(x) \in M = B_{d'}(f(a);\epsilon)$. The other way of writing this is: if $d(a,x) < \delta$ then $d'(f(a), f(x)) < \epsilon$. Therefore, f is continuous at $a \in X$. ∎

2.4 Limits

Recall that a sequence is just a function $a\colon \mathbb{Z}^+ \to (X,d)$. We want to discuss what happens to the sequence as we let n go to infinity; in other words what happens to the sequence as we look further and further into the range of a. Let us first recall the definitions in the real numbers and then try to set them up so that we can easily generalize them to arbitrary metric spaces.

Let $\{a_i\}$ be a sequence of real numbers. A real number L is said to be the *limit of the sequence* $\{a_n\}$ if, given any $\epsilon > 0$, there is a positive integer N such that whenever $n > N$, $|a_n - L| < \epsilon$. In this case we say that the sequence *converges to* L and write

$$\lim_{n\to\infty} a_n = L.$$

How can we generalize this to an arbitrary metric space? It should not be hard, because all we used in the definition was the distance function in the real numbers. We will use the distance function in our metric space similarly.

Definition 14 *Let $\{x_n\}$ be a sequence in the metric space (X,d). We say that this sequence $\{x_n\}$ converges to $x \in X$ if given any $\epsilon > 0$ there is a positive $N \in \mathbb{Z}^+$ so that whenever $n > N$, $d(x,x_n) < \epsilon$. In this case we will write* $\lim x_n = x$.

Lemma 2 *Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\lim x_n = x \in X$ if and only if for each neighborhood V of x there is an integer $N > 0$ so that $x_n \in V$ whenever $n > N$.*

This is nothing but applying the definitions of convergence and neighborhood, and its proof will be omitted.

If S is a set of infinite points and **there is at most a finite number** of elements of S for which a certain statement is false, thent he statement is said to be true for *almost all* of S. Thus, we may phrase the above lemma by saying that the sequence $\{x_n\}$ converges to x if each neighborhood of x contains almost all of the poins of the sequence.

One reason for looking a sequences is the concept of continuity. In calculus we define a function to be continuous at $a \in \mathbb{R}$ if the following conditions were met:

1. $\lim_{x\to a} f(x)$ exists;
2. $f(a)$ exists;
3. $\lim_{x\to a} f(x) = f(a)$.

It suffices to check this for all sequences appproaching a (a fact to be proven later). Thus, we find that we can show that f is continous at a if for each sequence $\{x_n\} \to a$, we have that $\{f(x_n)\} \to f(a)$.

We are able to extend this result to arbitrary metric spaces.

Theorem 11 *Let $f\colon (X,d) \to (Y,d')$. f is continous at a point $a \in X$ if and only if whenever $\lim x_n = x$ we have $\lim f(x_n) = f(x)$.*

The proof is straightforward.

PROOF: Assume that f is continuous and let $\{x_n\} \to x$ in X. Let $\epsilon > 0$ and let $M = B_{d'}(f(x);\epsilon) \in V$. There is a neighborhood U of x in X, such that $f(U) \subset M$. Since U is a neighborhood there is a $\delta > 0$ so that $B_d(x'\delta) \subset U$. Now, $\{x_n\} \to x$ so for this δ there is a positive integer N so that whenever $n > N$ we have $x_n \in B_d(x'\delta) \subset U$. Thus, $f(x_n) \in f(U) \subset M$. Therefore, for any neighborhood M of $f(x)$ there is a positive integer N so that whenever $n > N$ we have $d'(f(x), f(x_n) < \epsilon$, which implies that the sequence $\{f(x_n)\}$ converges to $f(x)$.

To prove the other direction, ∎

2.5 Open Sets and Closed Sets Revisited

Remember that we defined an open set as a set that is the union of open balls. A set is closed if its complement is open.

Theorem 12 *The open subsets of a metric space (X,d) have the following properties:*

1. *X and $\emptyset$ are open sets.*
2. *The union of any family of open sets is open.*
3. *The intersection of a finite family of open sets is open.*

PROOF: These are straightforward.

1. The whole space X is open since it is the union of all open balls with all possible centers and radii. The empty set is open since it is the union of the empty collection of open balls.

2. If $\{U_\alpha \mid \alpha \in A\}$ is a family of open sets in X, then each U_α is a union of open balls. Then $\bigcup_{\alpha\in A} U_\alpha$ is the union of all of the open balls that comprise each U_α and is hence open.

3. Let $\{U_i \mid i = 1, \ldots, n\}$ be a finite collection of open sets and let $x \in \bigcap_{i=1}^n U_i$. Then, by our previous theorem there exist ϵ_i, $i = 1, \ldots, n$ so that $B_d(x;\epsilon_i) \subset U_i$ and

$$\bigcap_{i=1}^{n} B_d(x;\epsilon_i) \subset \bigcap_{i=1}^{n} U_i.$$

   Let $\epsilon = \min\{\epsilon_i \mid i = 1, \ldots, n\}$. Then, $\bigcap_{i=1}^n B_d(x;\epsilon_i) = B_d(x;\epsilon)$. Thus, $B_d(x;\epsilon)$ is an open ball centered at x and contained in $\bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ∎

Theorem 13 *The closed subsets of a metric space (X, d) have the following properties:*

1. *X and $\emptyset$ are closed sets.*

2. *The intersection of any family of closed sets is closed.*

3. *The union of a finite family of closed sets is closed.*

This follows from our previous theorem and complements.

Definition 15 *Let (X, d) be a metric space and A a subset of X. A point $x \in X$ is a **limit point** or **accumulation point** of A provided that every open set containing x contains a point of A distinct from x. The set of limit points of A is called its **derived** set, denoted by A'.*

Lemma 3 *Let (X, d) be a metric space and A a subset of X. A point $x \in X$ is a limit point of A if and only if $d(x, A \setminus \{x\}) = 0$.*

Lemma 4 *A subset A of a metric space (X, d) is closed if and only if A contains all its limit points.*

PROOF: Let A be closed and let x be a limit point of A. If $x \notin A$ then $X \setminus A$ is an open set containing x but containing no other point of A. Thus, x could not be a limit point of A. This means that if x is a limit point of A, then it must be a member of A.

Now suppose that A contains all of its limit points. To show that A is closed, we must show that $X \setminus A$ is open. If $x \in X \setminus A$, then x is not a limit point of A. Thus, there is some open set U_x containing x but no other point of A. Then $X \setminus A$ is the union of all of these sets. Hence, $X \setminus A$ is open and A is closed. ∎

What is the connection between *limit points* and the *limit* of a sequence?

Theorem 14 *Let (X, d) be a metric space and A a subset of X.*

1. *A point $x \in X$ is a limit point of A if and only if there is a sequence of distinct points of A which converges to x.*

2. *The set A is closed if and only if each convergent sequence of points of A converges to a point of A.*

Corollary 1 *Let x be a limit point of a subset A of a metric space X. Then every open set containing x contains infinitely many members of A.*

2.6 Interior, Closure, and Boundary

Definition 16 *Let A be a subset of a metric space (X, d). A point $x \in A$ is an **interior point** of A if there is an open set U which contains x and is contained in A; $x \in U \subset A$. The **interior** of A, denoted intA, is the set of all interior points of A.*

Note that for the open set U in the definition, every point of U is an interior point of A. Thus, the interior of A contains every open set contained in A and is the union of this family of open sets. This means two things:

1. the interior of a set A is an open set, and

2. the interior of a set A is the largest open set contained in A.

Item (2) above means that if U is open and $U \subset A$, then $U \subset \text{int}A$.

Example 14

Let $X = \mathbb{R}$ with the usual metric.

1. For $a, b \in \mathbb{R}$ with $a < b$

$$\text{int}(a, b) = \text{int}[a, b) = \text{int}(a, b] = \text{int}[a, b] = (a, b).$$

2. The interior of a finite set is empty, since such a set cannot contain any open interval.

3. The interior of the set of irrational numbers is empty, since each open interval must contain some rational number. Likewise, the interior of the set of rationals is empty. If the rationals contained an open interval, then the set of rationals would have to be uncountable, since an open interval is uncountable.

4. $\text{int}\emptyset = \emptyset$; $\text{int}\mathbb{R} = \mathbb{R}$.

Definition 17 *The **closure** $\overline{A}$ of a subset of a metric space (X, d) is the union of the set A and the set of its limit points:*

$$\overline{A} = A \cup A'$$

where A' is the derived set of A.

Example 15

Let $X = \mathbb{R}$ with the usual metric.

1. For $a, b \in \mathbb{R}$ with $a < b$
$$\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b].$$

2. The closure of a finite set is itself, since the set of limit points of a finite set is empty.

3. The closure of the set of rational numbers is $\mathbb{R}$. Likewise, the closure of the set of irrationals is $\mathbb{R}$. Since every open interval contains both rational and irrational numbers.

4. $\overline{\emptyset} = \emptyset$; $\overline{\mathbb{R}} = \mathbb{R}$.

While the interior of a set is the largest open set contained in the set, the closure has a similar property described in the next theorem.

Theorem 15 *If $A \subset X$, then $\overline{A}$ is a closed set and is a subset of every closed set containing A.*

This says that the closure of a set is the smallest closed set containing the set.

Proof: To show that $\overline{A}$ is closed, we need to show that it contains all of its limit points. Suppose that $x \notin \overline{A}$. Then there is an open set U containing x so that $U \cap A = \emptyset$. Now, this means that U cannot contain a limit point of A either, since if an open set contains a limit point of A it must contain some other point of A also. Thus, U contains no point of $\overline{A}$, so x is not a limit point of $\overline{A}$. This means that all of the limit points of $\overline{A}$ must be contained in $\overline{A}$. Thus, $\overline{A}$ is closed.

Suppose now that F is a closed subset of X and $A \subset F$. Then we can show that $\overline{A} \subset \overline{F}$ and, since F contains all of its limit points, then $\overline{F} = F \cup F' = F$. Thus, $\overline{A} \subset F$ fore every closed set F containing A. ∎

Since this shows that $\overline{A}$ is the smallest closed set containing A, we can easily show that $\overline{A}$ is the intersection of all closed sets containing A.

Theorem 16 *Let A be a subset of the metric space (X, d).*

1. *A is open if and only if $A = int\ A$.*
2. *A is closed if and only if $A = \overline{A}$.*

Definition 18 *Let A be a subset of the metric space (X, d). A point $x \in X$ is a **boundary point** of A provided that $x \in \overline{A} \cap \overline{X \setminus A}$. The set of boundary points of A is called the **boundary** of A and is denoted by ∂A.*

The industrious reader will readily work to show that the following statements are equivalent for a subset A of X and a points x in the metric space (X, d).

1. $x \in \partial A$,
2. $x \in (\overline{A} \setminus \text{int } A)$,
3. Every open set containing x contains a point of A and a point of $X \setminus A$.
4. Every neighborhood of x contains a point of A and a point of $X \setminus A$.
5. $d(x, A) = d(x, X \setminus A) = 0$.
6. $x \in \overline{A} \cap \overline{X \setminus A}$.

Example 16 1. Let $X = \mathbb{R}$ with the usual metric. For $a, b \in \mathbb{R}$ with $a < b$

$$\partial(a, b) = \partial[a, b) = \partial(a, b] = \partial[a, b] = \{a, b\}.$$

2. In $\mathbb{R}^n$

$$\partial B(a; \epsilon) = \{x \in \mathbb{R}^n \mid d(a, x) = \epsilon\}.$$

3. The boundary of the set of all points in $\mathbb{R}^n$ having only rational coordinates is $\mathbb{R}^n$.

4. For any metric space (X, d),

$$\partial\emptyset = \partial X = \emptyset.$$

Chapter 3

Topological Spaces

3.1 Definition and Some Examples

We want to generalize the concepts that we developed in studying the metric spaces. We want to remove our reliance on a distance function. We were able to define most of what we wanted to do in metric spaces by defining our concepts in terms of the open sets. This was especially true of our study of continuous functions.

We will use the results that we proved about open sets as our basis for the generalization. We will define open sets as sets that satisfy certain conditions.

Definition 19 *Let X be a set and $\mathscr{T}$ a family of subsets of X satisfying the following properties.*

a) The set X and $\emptyset$ belong to $\mathscr{T}$,

b) The union of any family of members of $\mathscr{T}$ is a member of $\mathscr{T}$.

c) The intersection of any finite family of members of $\mathscr{T}$ is a member of $\mathscr{T}$.

Then $\mathscr{T}$ is called a ***topology*** *for X and the members of $\mathscr{T}$ are called* ***open sets****. The ordered pair $(X, \mathscr{T})$ is called a* ***topological space****, or simply a space.*

If we use the terminology *open sets* instead of *member of $\mathscr{T}$*, then the definition of a topological space may be restated as follows: *A family of subsets of X is a topology for X means that:*

a) Both X and $\emptyset$ are open sets.

b) The union of any family of open sets is an open set.

c) The intersection of any finite family of open sets is open.

Example 17 The *usual topology* for the real line $\mathbb{R}$ is the topology generated by its usual metric. We shall refer to the real line with the usual topology as simply the real line or $\mathbb{R}$.

Example 18 The *usual topology* on $\mathbb{R}^n$ is the topology generated by the usual metric on $\mathbb{R}^n$. It is also the topology generated by the taxicab metric and the max metric. Thus, the usual topology does not distinquish the metric determining it from the other two. We shall refer to $\mathbb{R}^n$ with the usual topology as *Euclidean n-space*, or simply $\mathbb{R}^n$.

Example 19 For any set X we take $\mathscr{T} = 2^X$ to be the set of all subsets of X. This clearly satisfies all of the properties of a topology, since we have included every possible subset in the topology. This is called the *discrete topology*. Note that it is the topology generated by the discrete metric. Also, note that this is the largest possible collection of open subsets of X.

Example 20 At the opposite extreme, we may take $\mathscr{T} = \{\emptyset, X\}$. This is called the *trivial topology*, or *indiscrete topology*, on X. This is the smallest collection of open sets on X.

Example 21 Let X be a set. We shall take $\mathscr{T}$ to consist of $\emptyset$, X, and all sets U so that $X \setminus U$ is a finite set. Then $\mathscr{T}$ is a topology on X called the *cofinite topology*, or *finite complement topology*. This is really of interest only when X is an infinite set. When X is a finite set, this is the same as the discrete topology.

Definition 20 *A subset F of a topological space X is* ***closed*** *if $X \setminus F$ is an open set.*

Theorem 17 *The closed sets of a topological space X have the following properties:*

a) X and $\emptyset$ are closed.

b) The intersection of any family of closed sets is closed.

c) The union of any finite family of closed sets is closed.

Definition 21 *Let $(X, \mathscr{T})$ be a topological space and let $A \subset X$. A point x in X is a* ***limit point*** *of A if every open set containing x contains a point of A distinct from x. The set of limit points of A is called the* ***derived set*** *of A, denoted A'.*

Example 22 Let $X = \{a, b, c, d\}$. Let $\mathscr{T}_0$ be the indiscrete topology; $\mathscr{T}_1$, the discrete topology; $\mathscr{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$; and
$\mathscr{T}_3 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The reader should verify that $\mathscr{T}_2$ and $\mathscr{T}_3$ are topologies on X. Let $A = \{a, b\}$, $B = \{c\}$, and $C = \{d\}$. We want to find the limit points of these sets in the different topologies.

	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$
A	X	$\emptyset$	$\{c, d\}$	$\{d\}$
B	$\{a, b, d\}$	$\emptyset$	$\{d\}$	$\{d\}$
C	$\{a, b, c\}$	$\emptyset$	$\{c\}$	$\emptyset$

Theorem 18 *A subset A of a topological space X is closed if and only if A contains all of its limit points.*

This is no surprise, and is proven exactly the way in which we proved it earlier in a metric space. We were careful there not to use the distance function, but to use the open sets.

Definition 22 *Let X be a topological space and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ **converges** to the point $x \in X$, or x is the **limit** of the sequence, if for each open set U containing x there is a positive integer N so that $x_n \in U$ for all $n \geq N$.*

Sequences are not as fundamental in general topological spaces as they are in metric spaces. The following example may show why.

Example 23 Consider $\mathbb{R}$ with the cofinite topology. Let $\{x_n\}$ be any sequence of real numbers. Let $a \in \mathbb{R}$ be ***any*** real number. Then $\{x_n\}$ converges to a, because if U is any open set containing a, then $\mathbb{R} \setminus U$ is a finite set. Since $\{x_n\}$ is an infinite set, we must have that infinitely many members of $\{x_n\}$ lie in U. Thus, there is a positive integer N such that if $n \geq N$ $x_n \in U$. Thus, $\{x_n\}$ converges to a. However, a was any arbitrary real number. This means that $\{x_n\}$ converges to ***every real number***. What is more (and maybe worse) is that $\{x_n\}$ was an arbitrary sequence. This means that every sequence converges to every real number. There are no non-convergent sequences and sequences do not have unique limits.

3.2 Interior, Closure and Boundary

Just as before we will define the interior, closure, and the boundary.

Definition 23 *Let A be a subset of the topological space X. A point $x \in A$ is an **interior point** of A if there is an open set U so that $x \in U \subset A$. A is called a **neighborhood** of x. The **interior** of A, denoted by A°, is the set of all interior points of A.*

*The **closure**, $\bar{A}$, of A is the union of A and it set of limit points:*

$$\bar{A} = A \cup A'.$$

*A point $x \in X$ is a **boundary point** of A if $x \in \bar{A} \cap \overline{X \setminus A}$. The set of boundary points of A is called the **boundary** of A and is denoted by ∂A.*

Theorem 19 *For any subsets A, B of a topological space X*

a) The interior of A is the union of all open sets contained in A and is the largest open set contained in A.

b) A is open if and only if $A = A^\circ$.

c) If $A \subset B$, then $A^\circ \subset B^\circ$.

d) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof: We will offer only a proof for (d). The others follow from closely from what we did in the case of a metric space.

Since $A \cap B$ is a subset of both A and B, then by (c) $(A \cap B)^\circ \subset A^\circ \cap B^\circ$. Now, $A^\circ \cap B^\circ$ is an open set and is a subset of $A \cap B$. Thus by (a), $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. This completes the proof. ∎

Theorem 20 *For any subsets A, B of a topological space X*

a) The closure of A is the intersection of all closed sets containing in A and is the smallest closed set containing in A.

b) A is closed if and only if $A = \overline{A}$.

c) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

d) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

We leave this to the reader to prove.

Theorem 21 *Let A be a subset of a topological space X.*

a) $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$.

b) ∂A, A°, and $(X \setminus A)^\circ$ are pairwise disjoint sets whose union is X.

c) ∂A is a closed set.

d) $\overline{A} = A^\circ \cup \partial A$.

e) A is open if and only if $\partial A \subset (X \setminus A)$.

f) A is closed if and only if $\partial A \subset A$.

g) A is open and closed if and only if $\partial A = \emptyset$.

PROOF: Parts (a)–(d) follow immediately from the definitions.

(e) If A is open then $A = A^\circ$. Now by (b) A° and ∂A are disjoint. Thus, A and ∂A are disjoint. This implies that $\partial A \subset X \setminus A$. Now, if $\partial A \subset X \setminus A$ then no point of A is a boundary point of A. Thus, every point of A is an interior point of A and $A = A^\circ$. Thus, A is open.

(f) This follows from our duality of open and closed sets.

(g) If A is both open and closed, then $\partial A \subset A \cap (X \setminus A) = \emptyset$. If $\partial A = \emptyset$ then clearly $\partial A \subset A$ — meaning A is closed — and $\partial A \subset (X \setminus A)$ — meaning A is open. ∎

Definition 24 *A set A in a topological space X is **dense** if $\overline{A} = X$. If X has a countable dense set, then X is a **separable** space.*

Example 24

1. The reals with the usual topology is separable, since the rationals are dense.
2. Euclidean n-space is separable, since the set of points having only rational coordinates is dense and countable.
3. The reals with the cofinite topology is separable, since every countable infinite set is dense.

Definition 25 *A subset B of a space X is **nowhere dense** if $(\overline{B})^\circ = \emptyset$.*

Note that a finite subset of a metric space is nowhere dense. In other topological spaces, we will see more interesting examples.

3.3 Basis for a Topology

It appears that a topology can be relatively large. In fact, for an infinite set the discrete topology consists of all subsets of the space, so it would be prohibitive to have to check all subsets. We have seen though that we can get by with just checking some of the sets. For the discrete topology we have usually only checked the singleton sets. For a metric space we were able to do everything we wanted by working with the open balls. In fact we defined all open sets in terms of the open balls. Can we do this in general? Can we find a certain collection of subsets that will generate all of the elements of the topology, just like the open balls generate the metric topology? The answer is yes, because we can take $\mathscr{T}$ as this generating set. This begs the answer, because we are looking for a *smaller* collection than the whole topology.

Definition 26 *Let $(X, \mathscr{T})$ be a topological space. A* ***basis*** *$\mathscr{B}$ for $\mathscr{T}$ is a subcollection of $\mathscr{T}$ with the property that each member of $\mathscr{T}$ is a union of members of $\mathscr{B}$. The members of $\mathscr{B}$ are called* ***basic open sets*** *and $\mathscr{T}$ is the topology* ***generated by $\mathscr{B}$.***

Example 25 Most of what we have seen is based on metric spaces.

1. The collection of all open intervals is a basis for the usual topology on the reals.
2. The collection of all open balls is a basis for the metric topology on the metric space (X, d).
3. For any set X the collection of all singleton sets $\{x\}$ is a basis for the discrete topology.

Definition 27 *Let $(X, \mathscr{T})$ be a topological space. A* ***local basis*** *at $a \in X$ is a subcollection $\mathscr{B}_a$ of $\mathscr{T}$ such that*

a) a belongs to each member of $\mathscr{B}_a$, and

b) each open set containing a contains a member of $\mathscr{B}_a$.

Definition 28 *A space X is* ***first countable*** *if there is a countable local basis at each point of X. The space X is* ***second countable*** *if the topology for X has a countable basis.*

Note that every second countable space is first countable because if there is a countable basis $\mathscr{B}$, then the number of these sets containing any given point $a \in X$ is at most countable.

Theorem 22 *Every second countable space is separable.*

PROOF: Let X be a second countable space with a countable basis $\mathscr{B}$. Let A be a set formed by choosing one element from each non-empty element of $\mathscr{B}$. Each point of X is a limit point of some point in A by the definition of a basis. Thus, A is dense in X. ∎

Theorem 23 *a) Every metric space is first countable.*

b) Every separable metric space is second countable.

The proof is left to the reader.

We have been starting with a topology and asking if there is a basis for it. We could be starting with a collection of open sets and asking if it forms a basis for a topology. Not every collection of open sets will work. When is a collection of open sets a basis for a topology on X?

Theorem 24 *A family $\mathscr{B}$ of subsets of a set X is a basis for a topology on X if and only if both of the following hold:*

a) The union of members of $\mathscr{B}$ is X.

b) For each $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there is a member B_x of $\mathscr{B}$ such that $x \in B_x \subset B_1 \cap B_2$.

Example 26 [Sorgenfrey Line] Let $\mathscr{B}$ be the collection of all half-open intervals of $\mathbb{R}$ of the form $[a, b)$, $a < b$. Clearly, the union of all of these intervals is $\mathbb{R}$. If we take two of these sets and intersect them, we can find another of these sets in the intersection. Thus, these sets form a basis for a topology on the real line, called the *half-open interval topology* $\mathscr{T}''$ for $\mathbb{R}$. $\mathbb{R}$ with this topology is called the *Sorgenfrey line*. The Sorgenfrey line has the property that it is first countable and separable, but not second countable.

3.4 Continuous Functions

We were able to move away from the epsilon-delta definition of continuity of a function in between two metric spaces by using open balls. We will use this as our starting point for general topological spaces.

Definition 29 *A function $f\colon (X, \mathscr{T}) \to (Y, \mathscr{T}')$ is* ***continuous*** *at a point $a \in X$ if for each open set V in Y containing $f(a)$ there is an open set $U \subset X$ containing a so that $f(U) \subset V$, or equivalently $U \subset f^{-1}(V)$.*

Now, we are more interested in the situation where the function is continous at every point of X. This means that for every open set V in Y and every point $a \in X$ with $f(a) \in V$, there is an open set $U_a \subset X$ with $a \in U_a$ and $f(U_a) \subset V$. Equivalently, we have $a \in U_a \subset f^{-1}(V)$. This means that for each point in $f^{-1}(V)$ we can find an open set containing that point and contained in $f^{-1}(V)$. Thus, $f^{-1}(V)$ must be open in X for each open set $V \subset Y$. This leads us to a more general definition.

Definition 30 *A function $f\colon (X, \mathscr{T}) \to (Y, \mathscr{T}')$ is* ***continuous*** *if for each open set V in Y $f^{-1}(V)$ is an open set in X.*

Theorem 25 *Let $f\colon X \to Y$ be a function on the topological spaces X and Y and let $a \in X$. The following are equivalent.*

a) f is continuous at a.

b) For each open set $V \in Y$ containing $f(a)$, there is an open set U in X such that $a \in U \subset f^{-1}(V)$.

c) For each neighborhood V of $f(a)$, $f^{-1}(V)$ is a neighborhood of a.

The proof is left to the reader.

Theorem 26 *Let $f\colon X \to Y$ be a function of topological spaces. The following are equivalent.*

(i) *f is continuous.*

(ii) *For each closed subset $C \subset Y$, $f^{-1}(C)$ is closed in X.*

(iii) *For each subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*

(iv) *There is a basis $\mathscr{B}$ for the topology of Y so that $f^{-1}(B)$ is open in X for each basic open set $B \in \mathscr{B}$.*

PROOF: To show that (*i*) implies (*ii*) will require the duality between open and closed sets. If $C \subset Y$ is closed, the $Y \setminus C$ is open in Y. Since f is continuous, $f^{-1}(Y \setminus C)$ is open in X. Hence $X \setminus (f^{-1}(Y \setminus C))$ is open. If $x \in X \setminus (f^{-1}(Y \setminus C))$ then $f(x) \notin Y \setminus C$, or $f(x) \in C$. Thus, $X \setminus (f^{-1}(Y \setminus C)) \subset f^{-1}(C)$. The opposite inclusion is clear. Thus, $f^{-1}(C) = X \setminus (f^{-1}(Y \setminus C))$ is closed. A similar analysis shows that (*ii*) implies (*i*).

To show that (*ii*) implies (*iii*), let $A \subset X$. Then $\overline{f(A)}$ is a closed subset of Y. Hence, $f^{-1}(\overline{f(A)})$ is a closed subset of X. Now, $A \subset f^{-1}(\overline{f(A)})$ so $\overline{A} \subset f^{-1}(\overline{f(A)})$. Thus, $f(\overline{A}) \subset \overline{f(A)}$.

To show that (*iii*) implies (*ii*), let C be a closed subset of Y. Then,

$$f(\overline{f^{-1}(C)}) \subset \overline{ff^{-1}(C)} \subset \overline{C} \subset C$$

so $\overline{f^{-1}(C)} \subset f^{-1}(C)$ making $f^{-1}(C)$ a closed set.

For the last equivalence, (*i*) clearly implies (*iv*). We need to prove the opposite implication. Let O be an open set in Y. Then by the definition of a basis, $O = \cup_{\alpha \in I} B_\alpha$ for some subcollection $\{B_\alpha\}_{\alpha \in I}$ of the basis $\mathscr{B}$. Then

$$f^{-1}(O) = f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha).$$

Since each $f^{-1}(B_\alpha)$ is open in X and the union of any family of open sets is open, then $f^{-1}(O)$ is open in X and f is continuous. ∎

Theorem 27 *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions, then $g \circ f\colon X \to Z$ is continuous.*

Definition 31 *A function $f\colon X \to Y$ is a* ***homeomorphism*** *if*

a) f is one-to-one, ***(injective)***

b) f is onto, ***(surjective)***

c) *f is continuous,*

d) *f^{-1} is continuous.*

Topological spaces are ***topologically equivalent*** *or* ***homeomorphic*** *if there homeomorphism from f from X onto Y.*

The continuity of the function **and** its inverse are extremely important! A property P of topological spaces is a ***topological property*** or ***topological invariant*** provided that if space X has property P, then so does every space which is homeomorphic to X.

Theorem 28 *Separability is a topological property.*

Theorem 29 *First countability and second countability are topological properties.*

Definition 32 *A topological space is* ***metrizable*** *provided that the topology on X is generated by a metric.*

Theorem 30 *Metrizability is a topological property.*

Since $\mathbb{R}$ and $(0,1)$ are homeomorphic, the property of being a bounded metric space is not a topological property. Likewise, distance is not a topological invariant.

3.5 Subspaces

Let $(X, \mathscr{T})$ be a topological space and let A be a subset of X. The ***relative topology*** or ***subspace topology*** $\mathscr{T}'$ on A determined by $\mathscr{T}$ consists of all sets of the form $O \cap A$ for which O is an open set of $\mathscr{T}$.

$$\mathscr{T}' = \{O \cap A \mid O \in \mathscr{T}\}.$$

The members of $\mathscr{T}'$ are called ***relatively open sets*** in A, and $(A, \mathscr{T}')$ is called a ***subspace*** of $(X, \mathscr{T})$.

Note that this is actually a topology for A.

$$\emptyset = \emptyset \cap A \qquad A = X \cap A,$$

so both $\emptyset$ and A are open in A. If $\{U_\alpha\}$ are open in A, then $U_\alpha = O_\alpha \cap A$ and

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha (O_\alpha \cap A) = \left(\bigcup_a lphaO_\alpha \right) \cap A$$

is relatively open since the union of any family of open sets is open in X. For any finite family of open sets $U_i = O_i \cap A$, we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (O_i \cap A) = \left(\bigcap_{i=1}^n O_i \right) \cap A$$

is relatively open since the intersection of any finite family of open sets is open in X.

Lemma 5 *Let $(a, \mathscr{T}')$ be a subspace of the topological space $(X, \mathscr{T})$. A subset F of A is closed in the subspace topology on A if and only if $F = C \cap A$ for some closed subset C of X.*

Example 27

1. The closed interval $[a, b]$ with $a < b$ is a subspace of $\mathbb{R}$ with the usual topology. The open sets containing a are sets of the form $[a, c)$ with $a < c < b$.
2. The subset of $\mathbb{R}^{n+1}$ consisting of all $(n+1)$-tuples $(x_1, x_2, \ldots, x_n, x_{n+1})$ with $x_{n+1} = 0$ is homeomorphic to $\mathbb{R}^n$
3. Let $a < b < c < d$. Let $A = [a, b] \cup (c, d)$ be considered as a subspace of the real line. Then the subset $[a, b]$ of A is both relatively open and relatively closed. It is clearly closed because $[a, b] = [a, b] \cap A$ and $[a, b]$ is closed in the real line. It is open because for $0 < \epsilon < c - b$, $[a, b] = (a - \epsilon, b + \epsilon) \cap Y$. Thus, we see that since (c, d) is the complement of this set that is both relatively open and relatively closed, then we see that (c, d) is both relatively open and relatively closed.

Definition 33 *A property P to topological spaces is* ***hereditary*** *provided that if X has property P, then every subspace of X has this property.*

Example 28

1. First countability and second countability are hereditary properties. If X has a countable basis, then intersecting these basis elements with A will give a countable basis for the subspace topology. First countable is similar.
2. Separability is not hereditary. Let $A \subset \mathbb{R}^2$ consist of the x-axis and the point $a = (0, 1)$. Define a topology $\mathscr{T}$ on A by taking the empty set and all subsets of A that contain the singleton set $\{a\}$. Then $(X, \mathscr{T})$ is separable because the singleton set $\{a\}$ is dense. Every point except a is a limit point of $\{a\}$. However, the subspace topology on $\mathbb{R}$ (the x-axis) is the discrete topology, so $(\mathbb{R}, \mathscr{T}')$ is not separable.

3.6 Hausdorff Spaces

A topological space X is a ***Hausdorff space*** if for each pair of distinct points $a, b \in X$ there exist disjoint open sets U and V such that $a \in U$ and $b \in V$.

Example 29

1. Every metric space is Hausdorff. We proved this as a homework problem, but it is simple. Let $r = d(a, b)$ and then take two open balls of radius $r/2$ centered at a and b respectively.
2. The real line with the co-finite topology is not Hausdorff. Likewise, the real line with the co-countable topology is not Hausdorff.

3. Take any set with more than one point and give it the indiscrete (trivial) topology. It is not Hausdorff.

4. The space in Example 2 is not Hausdorff, because you can never separate a from any other point.

5. [**The Zariski Topology**] Let n be a positive integer and consider the family $\mathscr{P}$ of all polynomials in n real variables $x_1, x_2, \ldots, x_n$. For $p \in \mathscr{P}$ let $Z(p)$ denote its solution set in $\mathbb{R}^n$:

$$Z(p) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid p(x_1, x_2, \ldots, x_n) = 0\}.$$

Let $\mathscr{B}$ be the collection of sets that are complements of some $Z(p)$ for some $p \in \mathscr{P}$. This forms a basis for a topology on $\mathbb{R}^n$ called the *Zariski topology.*

For the real line, $n = 1$, this is just the co-finite topology. This is because each finite set of real numbers is the solution set for some polynomial in one real variable. If $A = \{a_1, a_2, \ldots, a_n\}$, then

$$p(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$$

is a polynomial with A as its solution set. Likewise, the set of solutions of a polynomial in one real variable of dimension n is at most n.

For $n > 1$ this is not the co-finite topology. For example, the line $y = a$ in $\mathbb{R}^2$ is the solution set to the polynomial in two variables

$$p(x, y) = y - a.$$

Note that this is not a finite set. However, each finite set can serve as the solution set of a polynomial.

Now, $\mathbb{R}^n$ with the Zariski topology is not Hausdorff. Assume that $P = (a_1, a_2, \ldots, a_n)$ and $Q = (b_1, b_2, \ldots, b_n)$ are two distinct points in $\mathbb{R}^n$.

Theorem 31 1. *The property of being a Hausdorff space is topological and hereditary.*

2. *In a Hausdorff space a sequence $\{x_n\}_{n=1}^{\infty}$ cannot converge to more than one point.*

PROOF: We will prove 1 only and leave the other to the reader.

Suppose that X is Hausdorff and $f: X \to Y$ is a homeomorphism. For $a \neq b \in Y$, we have that $f^{-1}(a)$ and $f^{-1}(b)$ are distinct points in X. Thus, there are disjoint open sets $U, V \subset X$ so that $f^{-1}(a) \in U$ and $f^{-1}(b) \in V$. Hence, $f(U)$ and $f(V)$ are disjoint open sets of Y containing a and b respectively.

To show that Hausdorff is hereditary, assume that X is Hausdorff and that $A \subset X$. Let $a \neq b \in A$. Then, $a \neq b \in X$ and there are disjoint open sets $U, V \subset X$ so that $a \in U$ and $b \in V$. Then, $U \cap A$ and $V \cap A$ are disjoint relatively open subsets of A containing a and b respectively. ∎

Chapter 4

Connectedness

4.1 Connected and Disconnected Spaces

It is easier to define *disconnected* than *connected*, and we will do so.

Definition 34 *A topological space X is* ***disconnected*** *or* ***separated*** *if it is the union of two disjoint, non-empty open sets. Such a pair A, B of subsets of X is called a* ***separation*** *of X. A space is* ***connected*** *if it is not disconnected.*

A subspace Y of X is ***connected*** *provided that it is a connected space when assigned the subspace topology.*

Example 30

1. A discrete space with more than one point is disconnected.

2. Any set with the indiscrete topology is connected, since there do not exist two non-empty open sets.

3. Let A be the set of non-zero real numbers with the subspace topology. Then A is disconnected since $(-\infty, 0)$ and $(0, \infty)$ form a separation.

4. Let $B = \mathbb{R}^2 \setminus \mathbb{R}$ be the plane minus the real axis. B is disconnected.

5. $X = [0, 1] \cup [2, 3]$ is disconnected.

6. Let X be the set of real numbers with the addition of a point $(0, 1)$, and let the topology $\mathscr{T}$ for X consist of $\emptyset$, X, and all subsets of X which contain a. Now, X is connected because every open set contains the point a. On the other hand, as a subspace of $(X, \mathscr{T})$ the real line is assigned the discrete topology and is disconnected.

These last few examples show that the property of being connected is not hereditary.

Example 31 The real line $\mathbb{R}$ with the usual topology is connected. Suppose otherwise that

$$\mathbb{R} = A \cup B$$

where $A \cap B = \emptyset$ are both open and non-empty. Since

$$A = \mathbb{R} \setminus B \quad \text{and} \quad B = \mathbb{R} \setminus A$$

we see that A and B are also both closed. Consider two points $a, b \in \mathbb{R}$ with $a \in A$ and $b \in B$. We may assume that $a < b$.

Put $A^+ = A \cap [a, b]$. Then A^+ is a closed and bounded subset of $\mathbb{R}$. Thus it must contain its least upper bound c. Now, $c \neq b$ since A and B have no points in common. Thus, $c < b$. Thus, A contains no points of $(c, b]$, placing $(c, b] \subset B$. Thus, $c \in \overline{B}$. Now, B is closed, so $c \in B$. This implies that $c \in A \cap B$, contradicting the assumption that A and B are disjoint. Thus, $\mathbb{R}$ is connected.

4.2 How to Tell If a Space is Connected?

Definition 35 *Non-empty subsets A and B are* ***separated sets*** *if* $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Theorem 32 *The following are equivalent for a topological space X.*

1. *X is disconnected.*
2. *X is the union of two disjoint, non-empty closed sets.*
3. *X is the union of two separated sets.*
4. *There is a continuous function from X onto the discrete two-point space $\{0, 1\}$.*
5. *X has a proper subset A which is both open and closed.*
6. *X has a proper subset A such that*

$$\overline{A} \cap \overline{(X \setminus A)} = \emptyset.$$

PROOF: First, we will show that (1) $\Leftrightarrow$ (2). Assume that X is disconnected. Then there are two nonempty, disjoint open sets A and B, so that $X = A \cup B$. This means that $X \setminus A = B$ and $X \setminus B = A$. Since both A and B are open, we now have that both A and B are closed, and the result follows. The proof of the opposite implication is analogous.

(1) $\Leftrightarrow$ (3): Assume that X is disconnected. We have then that there are two nonempty, disjoint open sets A and B, so that $X = A \cup B$. We just proved that A and B are closed. Thus, $\overline{A} = A$ and $\overline{B} = B$. Hence,

$$\overline{A} \cap B = A \cap B = \emptyset = A \cap B = A \cap \overline{B}.$$

Again, the opposite implication is analogous.
(1) ⇔ (4): Assume that X is disconnected. Define a function $f\colon X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B. \end{cases}$$

f is clearly onto. Since we have placed the discrete topology on $\{0,1\}$, we only need to check $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ to check that f is continuous. Since $f^{-1}(\{0\}) = A$ is open and $f^{-1}(\{1\}) = B$ is open, we see that f is continous.

Assume that there is a continous, onto function $f\colon X \to \{0,1\}$. Since $\{0,1\}$ has the discrete topology, the sets $\{0\}$ and $\{1\}$ are open and disjoint. Let $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. These then form a separation for X.
(1) ⇔ (5): Assume that X is disconnected and that A and B form a separation of X. Then A is a proper subset which is both open and closed. If $U \subset X$ is a proper subset which is both open and closed, then $V = X \setminus U$ is open and U and V form a separation of X.
(1) ⇔ (6): Assume that X is disconnected and A and B form a separation of X. Then $X \setminus A = B$, $\overline{A} = A$, and $\overline{(X \setminus A)} = \overline{B} = B = X \setminus A$. Hence

$$\overline{A} \cap \overline{(X \setminus A)} = \emptyset.$$

If such a set A exists, then $\overline{A}$ and $\overline{X \setminus A}$ form a separation of X. ∎

Corollary 2 *The following statements are equivalent.*

1. *X is connnected.*
2. *X is not the union of two disjoint, non-empty closed sets.*
3. *X is not the union of two separated sets.*
4. *There is not continuous function from X onto a discrete two point space $\{0,1\}$.*
5. *The only subsets of X that are both open and closed are X and $\emptyset$.*
6. *X has no proper subset A so that*

$$\overline{A} \cap \overline{(X \setminus A)} = \emptyset.$$

Theorem 33 *Let X be a connected topological space and let $f\colon X \to Y$ be a continuous surjective function. Then Y is a connected space.*

PROOF: We will prove that if Y is not connected, then X is not connected, completing the proof by proving the contrapositive. If Y is disconnected, let A and B form the separation of Y. Then $Y = A \cup B$ and the sets $f{-}1(A)$ and $f^{-1}(B)$

(a) are open sets since f is continous;

(b) are disjoint since f is a function;

(c) are non-empty since f is surjective; and

(d) have union X because

$$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

Thus, if Y is disconnected, then X is disconnected. ∎

Corollary 3 *If $f\colon X \to Y$ is a continuous function, then $f(X)$, the image of f, is connected.*

Lemma 6 *A subspace Y of a space X is disconnected if and only if there are open sets U and V in X such that*

$$U \cap Y \neq \emptyset, \quad V \cap Y \neq \emptyset, \quad U \cap V \cap Y = \emptyset, \quad Y \subset U \cup V.$$

Theorem 34 *If Y is a connected subspace of X, then $\overline{Y}$ is connected.*

PROOF: Suppose that Y is connected. Consider a continuous function $f\colon \overline{Y} \to \{0,1\}$. We must show that f is not surjective. We know that the restriction $f|_Y$ is not surjective, so that f maps Y either to $\{0\}$ or $\{1\}$. Assume that $f(Y) = \{0\}$. Since f is continuous, we know that

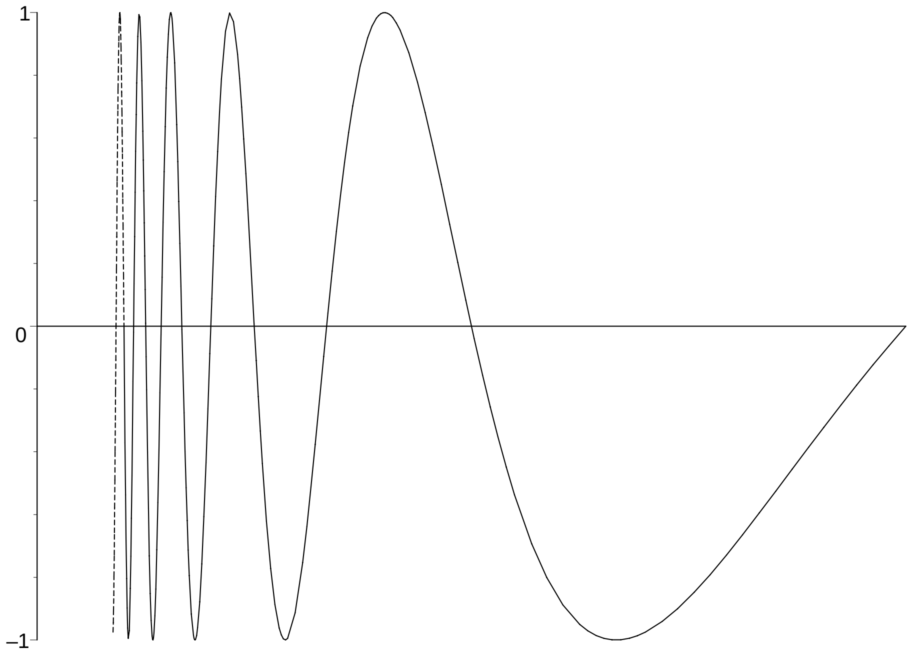
$$f(\overline{Y}) \subset \overline{(f(Y))} = \overline{\{0\}} = \{0\},$$

so f is not surjective. Thus, $\overline{Y}$ is connected. ∎

Corollary 4 *Let Y be a connected subspace of X and Z a subspace so that $Y \subset Z \subset \overline{Y}$. Then Z is connected.*

Example 32 Each interval of the real line is connected. We know that each open interval of the real line is connected, being homeomorphic to the real line. A non-degenerate closed interval is the closure of an open interval, and is hence connected. A degenerate closed interval is a single point, which is connected. Any other interval is trapped between an open interval and its closure, so it is connected. The empty set, which is an interval, is connected because there are no non-empty subsets.

Example 33 [**The Topologist's Sine Curve**] Let $A = \{(0, y) \mid -1 \leq y \leq 1\}$ and $B = \{(x, y) \mid 0 < x \leq 1,\ y = \sin(\pi/x)\}$. Put $T = A \cup B$. T is called the *topologist's sine curve*. Note that B is connected, since it is the continuous image of $(0, 1]$. Note also that $T = \overline{B}$, so that T is connected.



We should never believe that the union of two connected sets is connected, since one of our first examples of a disconnected space was two disjoint intervals. What would it take for a union of connected sets to be connected?

Theorem 35 *Let X be a space and let $\{A_\alpha \mid \alpha \in I\}$ be a family of connected subsets of X for which $\cap_{\alpha \in I} A_\alpha$ is not empty. Then $\cup_{\alpha \in I} A_\alpha$ is connected.*

PROOF: We shall use Lemma 6 to prove that $Y = \cup_{\alpha \in I} A_\alpha$ is connected. Suppose that U and V are open sets of X so that:

$$U \cap Y \neq \emptyset, \qquad U \cap V \cap Y = \emptyset, \qquad Y \subset U \cup V.$$

We need to then show that $Y \cap V = \emptyset$. This will show that Y is connected. Since $U \cap Y \neq \emptyset$, U must contain some point of some A_β, for some $\beta \in I$. Since A_β is connected, then $A_\beta \subset U$. If $b \in \cap_{\alpha \in I} A_\alpha$, then b must be in A_β, so $b \in U$. Thus, U contains a point b in each A_α, $\alpha \in I$. Since A_α is connected, the $A_\alpha \subset U$ for each $\alpha \in I$. Thus,

$$Y = \bigcup_{\alpha \in I} A_\alpha \subset U$$

so $V \cap Y = \emptyset$. ∎

Various variants of this theorem are as follows:

Lemma 7 *Let X be a space, $\{A_\alpha \mid \alpha \in I\}$ a family of connected subsets of X, and B a connected subset of X such that for each $\alpha \in I$, $A_\alpha \cap B \neq \emptyset$. Then $B \cup (\cup_{\alpha \in I} A_\alpha)$ is connected.*

Lemma 8 *Let $\{A_n\}$ be a sequence of connected subsets of a space X such that for each integer $n \geq 1$, A_n has at least one point in common with one of the preceding sets $A_1, \ldots, A_{n-1}$. Then $\cup_{n=1}^{\infty} A_n$ is connected.*

Definition 36 *A* ***connected component****, or* ***component****, of a topological space is a connected subset C of X which is not a proper subset of any connected subset of X.*

Theorem 36 *For a topological space X*

1. *Each point $x \in X$ belongs to exactly one component. The component C_x containing x is the union of all the connected subsets of X which contain x, and is thus the largest connected subset of X containing the point x.*
2. *For points $x, y \in X$ the components C_x and C_y are either disjoint or identical.*
3. *Every connected subset of X is contained in a component.*
4. *Each component is a closed set.*
5. *X is connected if and only if it has one component.*
6. *If C is a component of X and A and B form a separation for X, then C is a subset of A or a subset of B.*

Example 34

1. For the subspace $X = (0,1) \cup (2,3)$ of $\mathbb{R}$, there are two components, $(0,1)$ and $(2,3)$. Both components are closed sets with respect to the subspace topology.
2. In a discrete space, each component consists of only one point.
3. For the set $\mathcal{Q}$ of rational numbers with the subspace topology, each component consists of only one point. Note, however, that the subspace topology is not the discrete topology.

Example 35 Let X be the subspace of $\mathbb{R}^2$ consisting of the following sequence:

$$X = \{(x,y) \mid 0 \leq x \leq 1, \quad y = \frac{1}{n}, n = 1, \ldots, \infty\} \bigcup \left(\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \right)$$

Now, $[0, \frac{1}{2})$ is the component of X containing 0 and $(\frac{1}{2}, 1]$ is the component of X containing 1. Thus, 0 and 1 belong to different components. However, for any separation of X into disjoint non-empty open sets A and B whose union is X, both 0 and 1 belong to A or to B.

Definition 37 *A space X is* ***totally disconnected*** *if each component of X consists of a single point.*

Thus, a discrete space is totally disconnected, as is the subspace of rational numbers in the real line.

4.3 Applications of Connectedness

You have been using the properties of connectedness in Calculus without knowing it. The *Intermediate Value Theorem* as well as several fixed point theorems all depend on connectedness.

Theorem 37 (Intermediate Value Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on a closed interval, $[a,b]$. Let y_0 be a real number between $f(a)$ and $f(b)$. Then there is number $c \in [a,b]$ for which $f(c) = y_0$.*

PROOF: The interval $[a,b]$ is connected, so $f([a,b])$ is connected, since f is continuous. Thus, $f([a,b])$ is an interval in $\mathbb{R}$. Therefore, any number, y_0, between $f(a)$ and $f(b)$ must be in the image $f([a,b])$. This means that $y_0 = f(c)$ for some $c \in [a,b]$. ∎

You have used the following corollary numerous times.

Corollary 5 *Let $f\colon [a,b] \to \mathbb{R}$ be continuous for which one of $f(a)$ and $f(b)$ is positive and the other is negative. Then the equation $f(x) = 0$ has at least one root between a and b.*

Theorem 38 (Fixed Point Theorem) *Let $f\colon [a,b] \to [a,b]$ be a continuous function on the closed interval $[a,b]$. Then there is a $c \in [a,b]$ so that $f(c) = c$.*

PROOF: If $f(a) = a$ or if $f(b) = b$ then we are done, so assume that $f(a) \neq a$ and $f(b) \neq b$. Thus, $a < f(a)$ and $f(b) < b$ since the range of f is $[a,b]$. Define a new function $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - x, \qquad x \in [a,b].$$

Then $g(a) = f(a) - a > 0$ and $f(b) = f(b) - b < 0$. and g is continuous. Thus, by the corollary, there is a $c \in [a,b]$ so that $g(c) = 0$ or $f(c) - c = 0$. Thus, there is a $c \in [a,b]$ so that $f(c0 = c$. ∎

If $f\colon X \to X$ is a function, then we say that x is a *fixed point* of f if $f(x) = x$. A topological space X has the *fixed point property* if every continuous function $f\colon X \to X$ has at least one fixed point. We can restate the above theorem as follows:

Theorem 39 *Every closed and bounded interval has the fixed-point property.*

Theorem 40 *The fixed-point property is a topological invariant.*

Example 36 The real line does not have the fixed-point property. An example is $f(x) = x = 1$. Thus, no open interval has the fixed-point property. The intervals of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, and $[a, \infty)$ also do not have the fixed point property.

The n-sphere, S^n, $n \geq 1$, does not have the fixed-point property since the function $g(x) = -x$ has no fixed point.

4.4 Path-Connected Spaces

Our initial idea of connectedness is really best summed up in the definition of path connected. We usually think of a space as being connected if we can get from any one point to another without leaving the space. This is a stronger property than connectedness, though.

We use I to denote the unit interval, $[0, 1]$.

Definition 38 *A* ***path*** *in a space X is a continuous function $f\colon I \to X$. The points $f(0)$ and $f(1)$ are the* ***endpoints*** *of the path. The path f is called a* ***path from*** *$f(0)$* ***to*** *$f(1)$. If f is a path in X, the* ***reverse path*** *$\overline{f}$ is the path defined by*

$$\overline{f}(t) = f(1 - t), \quad t \in I.$$

Definition 39 *A space X is* ***path connected*** *if for any pair of points $x, y \in X$ there is a path in X with initial point x and terminal point y. A subspace A of X is* ***path connected*** *provided that A is path connected with its subspace topology.*

Example 37 Every interval on the real line is path connected. For $a, b \in K$ define the path by

$$p(t) = a(1 - t) + bt, \qquad t \in I.$$

Example 38 The generalizations of the interval to subsets of $\mathbb{R}^n$ are called *convex sets*. A set C of $\mathbb{R}^n$ is convex if for any two points, $a, b \in C$, the line segment joining a and b lies entirely in C. Since a line segment defines a path, each convex set is path connected.

Theorem 41 *Every path connected set is connected.*

PROOF: Suppose that X is path connected and let $a \in X$. For each $x \in X$ let C_x denote the image of the path from a to x. Since each C_x is connected and $a \in C_x$ for all $x \in X$, then by our previous theorem

$$X = \bigcup_{x \in X} C_x$$

is connected. ∎

Example 39 The Topologist's Sine Curve, T, is connected but it is not path connnected.

Thus, while in $\mathbb{R}$ connected and path connected are identical, in $\mathbb{R}^n$ there are sets that are connected, but not path connected. All is not lost though.

Theorem 42 *Every open, connected subset of $\mathbb{R}^n$ is path connected.*

There are analogous results for path connectedness as for connectedness. Likewise, one defines the ***path component*** of a space X as a path connected subset of X which is not a proper subset of any path connected subset of X.

4.5 Locally Connected and Locally Path Connected Spaces

The terms connected and path connected are global properties, *i.e.*, properties that apply to the whole space. *Local topological properties* are characteristics of a space "near" a particular point.

Definition 40 *A topological space X is **locally connected** at a point $p \in X$ if every open set containing p also contains a connected set which contains p. The space X is locally connected if it is locally connected at each point.*

Theorem 43 *Let X be a topological space.*

a) X is locally connected at a point $p \in X$ if and only if there is a local basis at p consisting of connected open sets.

b) X is locally connected if and only if it has a basis of connected open sets.

Example 40

1. Any interval in $\mathbb{R}$ is connected and locally connected.
2. $\mathbb{R}^n$ is connected and locally connected for each integer $n \geq 0$.
3. The subspace $[0,1] \cup [2,3]$ is locally connected, but not connected.
4. Let $X = \{(x,0) \mid 0 \leq x \leq 1\}$, $Y = \{(0,y) \mid 0 \leq y \leq 1\}$, and $Z = \left\{ \left(\frac{1}{n}, y\right) \middle| n \in \mathbb{Z}^+,\ 0 \leq y \leq 1 \right\}$. Let $C = X \cup Y \cup Z$. C is called the *topologist's comb*.

   The topologist's comb is obviously connected, being path connected. A path can run up and down the vertical tines and across the base. However, C is not locally connected at any point $(0,t)$, $0 < t \leq 1$, since small open sets containing such points consist of collections of open vertical intervals.
5. The set $\mathbb{Q}$ of rational numbers is neither connected nor locally connected.

Thus, from these examples, we see that one property does not imply the other. This is commonly true with global and local properties, but even this statement is not infallible. Recall that second countability does imply first countability.

Theorem 44 *A space X is locally connected if and only if for each open subset $O \subset X$, each component of O is an open set.*

Definition 41 *A space X is **locally path connected** at a point $p \in X$ if every open set containing p contains a path connected set containing p. The space X is **locally path connected** if it is locally path connected at each point.*

Theorem 45 *Let X be a topological space.*

a) X is locally path connected at a point $p \in X$ if and only if there is a local basis at p consisting of path connected open sets.

b) X is locally path connected if and only if it has a basis of path connected open sets.

Theorem 46 *A space X is locally path connected if and only if for each open subset $O \subset X$, each path component of O is an open set.*

Theorem 47 *If X is a connected, locally path connected space, then X is path connected.*

PROOF: For each point $x \in X$, let P_x denote the path component of X to which x belongs. Since X is an open set, Theorem 46 shows that each P_x is open. Recall that path components are either disjoint or identical.

For a particular point $a \in X$, suppose that $P_a \neq X$. Then P_a and the union of all P_x for which $x \notin P_a$ are disjoint, non-empty open subsets of X whose union is X. This would imply that X is disconnected. Since X is connected, we have that $P_a = X$ and X must be path connected as well. ∎

Chapter 5

Compactness

Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line: the Heine-Borel Property. While compact may infer "small" size, this is not true in general. We will show that $[0, 1]$ is compact while $(0, 1)$ is not compact.

Compactness was introduced into topology with the intention of generalizing the properties of the closed and bounded subsets of $\mathbb{R}^n$.

5.1 Compact Spaces and Subspaces

Definition 42 *Let A be a subset of the topological space X. An* ***open cover*** *for A is a collection $\mathscr{O}$ of open sets whose union contains A. A* ***subcover*** *derived from the open cover $\mathscr{O}$ is a subcollection $\mathscr{O}'$ of $\mathscr{O}$ whose union contains A.*

Example 41 Let $A = [0, 5]$ and consider the open cover

$$\mathscr{O} = \{(n-1, n+1) \mid n = -\infty, \dots, \infty\}.$$

Consider the subcover $\mathscr{P} = \{(-1, 1), (0, 2), (1, 3), (2, 4), (3, 5), (4, 6)\}$ is a subcover of A, and happens to be the smallest subcover of $\mathscr{O}$ that covers A.

Definition 43 *A topological space X is* ***compact*** *provided that every open cover of X has a finite subcover.*

This says that however we write X as a union of open sets, there is always a finite subcollection $\{O_i\}_{i=1}^n$ of these sets whose union is X. A subspace A of X is ***compact*** if A is a compact space in its subspace topology. Since relatively open sets in the subspace topology are the intersections of open sets in X with the subspace A, the definition of compactness for subspaces can be restated as follows.

Alternate Definition: *A subspace A of X is* ***compact*** *if and only if every open cover of A by open sets in X has a finite subcover.*

Example 42 1. Any space consisting of a finite number of points is compact.

2. The real line $\mathbb{R}$ with the finite complement topology is compact.

3. An infinite set X with the discrete topology is not compact.

4. The open interval $(0,1)$ is not compact. $\mathscr{O} = \{(1/n, 1) \mid n = 2, \ldots, \infty\}$ is an open cover of $(0,1)$. However, no finite subcollection of these sets will cover $(0,1)$.

5. $\mathbb{R}^n$ is not compact for any positive integer n, since $\mathscr{O} = \{B(\mathbf{0}, n) \mid n = 1, \ldots, \infty\}$ is an open cover with no finite subcover.

A sequence of sets $\{S_n\}_{n-1}^{\infty}$ is **nested** if $S_{n+1} \subset S_n$ for each positive integer n.

Theorem 48 (Cantor's Nested Intervals Theorem) *If $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a nested sequence of closed and bounded intervals, then $\cap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$. If, in addition, the diameters of the intervals converge to zero, then the intersection consists of precisely one point.*

PROOF: Since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for each $n \in \mathbb{Z}^+$, the sequences $\{a_n\}$ and $\{b_n\}$ of left and right endpoints have the following properties:

(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \ldots$ and $\{a_n\}$ is an increasing sequence;

(ii) $b_1 \geq b_2 \geq \cdots \geq b_n \geq \ldots$ and $\{b_n\}$ is a decreasing sequence;

(iii) each left endpoint is less than or equal to each right endpoint.

Let c denote the least upper bound of the left endpoints and d the greatest lower bound of the right endpoints. The existence of c and d are guaranteed by the Least Upper Bound Property. Now, by property (iii), $c \leq b_n$ for all n, so $c \leq d$. Since $a_n \leq c \leq d \leq b_n$, then $[c, d] \subset [a_n, b_n]$ for all n. Thus, $\cap_{n=1}^{\infty}[a_n, b_n]$ contains the closed interval $[c, d]$ and is thus non-empty.

If the diameters of $[a_n, b_n]$ go to zero, then we must have that $c = d$ and c is the one point of the intersection. ∎

Theorem 49 *The interval $[0,1]$ is compact.*

PROOF: Let $\mathscr{O}$ be an open cover. Assume that $[0,1]$ is not compact. Then either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ is not covered by a finite number of members of $\mathscr{O}$. Let $[a_1, b_1]$ be the half that is not covered by a finite number of members of $\mathscr{O}$.

Apply the same reasoning to the interval $[a_1, b_1]$. One of the halves, which we will call $[a_2, b_2]$, is not finitely coverable by $\mathscr{O}$ and has length $\frac{1}{4}$. We can continue this reasoning inductively to create a nested sequence of closed intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$, none of which is finitely coverable by $\mathscr{O}$. Also, by construction, we have that

$$b_n - a_n = \frac{1}{2^n},$$

so the diameters of these intervals goes to zero.

By the Cantor Nested Intervals Theorem, we know that there is precisely one point in the intersection of all of these intervals; $p \in [a_n, b_n]$, for all n. Since $p \in [0,1]$ there is an open interval $O \in \mathscr{O}$ with $p \in O$. Thus, there is a positive number, $\epsilon > 0$ so that $(p-\epsilon, p+\epsilon) \subset O$. Let N be a positive integer so that $1/2^N < \epsilon$. Then since $p \in [a_N, b_n]$ it follows that

$$[a_n, b_n] \subset (p-\epsilon, p+\epsilon) \subset O.$$

This contradicts the fact that $[a_N, b_N]$ is not finitely coverable by $\mathscr{O}$ since we just covered it with one set from $\mathscr{O}$. This contradiction shows that $[0,1]$ is finitely coverable by $\mathscr{O}$ and is compact. ∎

Compactness is defined in terms of open sets. The duality between open and closed sets and if $C_\alpha = X \setminus O_\alpha$,

$$X \setminus \left(\bigcap_{\alpha \in I} C_\alpha \right) = \bigcup_{\alpha \in I} O_\alpha$$

leads us to believe that there is a characterization of compactness with closed sets.

Definition 44 *A family $\mathscr{A}$ of subsets of a space X has the* ***finite intersection property*** *provided that every finite subcollection of $\mathscr{A}$ has non-empty intersection.*

Theorem 50 *A space X is compact if and only if every family of closed sets in X with the finite intersection property has non-empty intersection.*

This says that if $\mathscr{F}$ is a family of closed sets with the finite intersection property, then we must have that $\bigcap_{\mathscr{F}} C_\alpha \neq \emptyset$.

PROOF: Assume that X is compact and let $\mathscr{F} = \{C_\alpha \mid \alpha \in I\}$ be a family of closed sets with the finite intersection property. We want to show that the intersection of all members of $\mathscr{F}$ is non-empty. Assume that the intersection is empty. Let $\mathscr{O} = \{O_\alpha = X \setminus C_\alpha \mid \alpha \in I\}$. $\mathscr{O}$ is a collection of open sets in X. Then,

$$\bigcup_{\alpha \in I} O_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha = X \setminus \bigcap_{\alpha \in I} C_\alpha = X \setminus \emptyset = X.$$

Thus, $\mathscr{O}$ is an open cover for X. Since X is compact, it must have a finite subcover; *i.e.*,

$$X = \bigcup_{i=1}^{n} O_{\alpha_i} = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} C_{\alpha_i}.$$

This means that $\cap_{i=1}^{n} C_{\alpha_i}$ must be empty, contradicting the fact that $\mathscr{F}$ has the finite intersection property. Thus, if $\mathscr{F}$ has the finite intersection property, then the intersection of all members of $\mathscr{F}$ must be non-empty.

The opposite implication is left as an exercise. ∎

Is compactness hereditary? No, because $(0,1)$ is not a compact subset of $[0,1]$. It is *closed hereditary.*

Theorem 51 *Each closed subset of a compact space is compact.*

PROOF: Let A be a closed subset of the compact space X and let $\mathscr{O}$ be an open cover of A by open sets in X. Since A is closed, then $X \setminus A$ is open and

$$\mathscr{O}^* = \mathscr{O} \cup \{X \setminus A\}$$

is an open cover of X. Since X is compact, it has a finite subcover, containing only finitely many members $O_1, \ldots, O_n$ of $\mathscr{O}$ and may contain $X \setminus A$. Since

$$X = (X \setminus A) \cup \bigcup_{i=1}^{n} O_i,$$

it follows that

$$A \subset \bigcup_{i=1}^{n} O_i$$

and A has a finite subcover. ∎

Is the opposite implication true? Is every compact subset of a space closed? Not necessarily. The following though is true.

Theorem 52 *Each compact subset of a Hausdorff space is closed.*

PROOF: Let A be a compact subset of the Hausdorff space X. To show that A is closed, we will show that its complement is open. Let $x \in X \setminus A$. Then for each $y \in A$ there are disjoint sets U_y and V_y with $x \in V_y$ and $y \in U_y$. The collection of open sets $\{U_y \mid y \in A\}$ forms an open cover of A. Since A is compact, this open cover has a finite subcover, $\{U_{y_i} \mid i = 1, \ldots, n\}$. Let

$$U = \bigcup_{i=1}^{n} U_{y_i} \qquad V = \bigcap_{i=1}^{n} V_{y_i}.$$

Since each U_{y_i} and V_{y_i} are disjoint, we have U and V are disjoint. Also, $A \subset U$ and $x \in V$. Thus, for each point $x \in X \setminus A$ we have found an open set, V, containing x which is disjoint from A. Thus, $X \setminus A$ is open, and A is closed. ∎

Corollary 6 *Let X be a compact Hausdorff space. A subset A of X is compact if and only if it is closed.*

The following results are left to the reader to prove.

Theorem 53 *If A and B are disjoint compact subsets of a Hausdorff space X, then there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.*

Corollary 7 *If A and B are disjoint closed subsets of a compact Hausdorff space X, then there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.*

5.2 Compactness and Continuity

Theorem 54 *Let X be a compact space and $f\colon X \to Y$ a continuous function from X onto Y. Then Y is compact.*

PROOF: We will outline this proof. Start with an open cover for Y. Use the continuity of f to pull it back to an open cover of X. Use compactness to extract a finite subcover for X, and then use the fact that f is onto to reconstruct a finite subcover for Y. ∎

Corollary 8 *Let X be a compact space and $f\colon X \to Y$ a continuous function. The image $f(X)$ of X in Y is a compact subspace of Y.*

Corollary 9 *Compactness is a topological invariant.*

Theorem 55 *Let X be a compact space, Y a Hausdorff space, and $f\colon X \to Y$ a continuous one-to-one function. Then f is a homeomorphism.*

5.3 Locally Compact and One-Point Compactifications

Is it always possible to consider a topological space as a subspace of a compact topological space? We can consider the real line as an open interval (they are homeomorphic). Can we always do something of this sort?

Definition 45 *A space X is **locally compact at a point** $x \in X$ provided that there is an open set U containing x for which $\overline{U}$ is compact. A space is **locally compact** if it is locally compact at each point.*

Note that every compact space is locally compact, since the whole space X satisfies the necessary condition. Also, note that locally compact is a topological property. However, locally compact does not imply compact, because the real line is locally compact, but not compact.

Definition 46 *Let X be a topological space and let ∞ denote an ideal point, called the **point at infinity**, not included in X. Let $X_\infty = X \cup \infty$ and define a topology $\mathscr{T}_\infty$ on X_∞ by specifying the following open sets:*

(a) the open sets of X, considered as subsets of X_∞;

(b) the subsets of X_∞ whose complements are closed, compact subsets of X; and

(c) the set X_∞.

*The space $(X_\infty, \mathscr{T}_\infty$ is called the **one point compactification** of X.*

Theorem 56 *Let X be a topological space and X_∞ its one-point compactification. Then*

a) X_∞ is compact.

b) $(X, \mathscr{T})$ is a subspace of $(X_\infty, \mathscr{T}_\infty)$.

c) X_∞ is Hausdorff if and only if X is Hausdorff and locally compact.

d) X is a dense subset of X_∞ if and only if X is not compact.

PROOF:

a) Any open cover $\mathscr{O}$ of X_∞ must have a member U containing ∞. Since the complement $X_\infty \setminus U$ is compact, it has a finite subcover $\{O_i\}_{i=1}^n$ derived from $\mathscr{O}$. Thus, $U, O_1, \ldots, O_n$ is a finite subcover of X_∞.

b) The fact that $(X, \mathscr{T})$ is the subspace topology in $(X_\infty, \mathscr{T}_\infty)$ basically follows from the definition of the extended topology. It also requires that we look at what open sets containing the point at infinity look like. One such set is $U = X_\infty$ itself and $U \cap X = X$ is open in X. The second type is a subset of X_∞ so that $X \setminus U$ is closed and compact in X. In this case $U \cap X$ is open since its complement is closed.

c) Suppose that X_∞ is Hausdorff. Then X is Hausdorff since the property is hereditary. Now, let $p \in X$. Since X_∞ is Hausdorff, there are open, disjoint sets U and V in X_∞ so that $\infty \in U$ and $p \in V$. Thus, $V \subset X_\infty \setminus U$ and this latter set is closed and compact in X. Hence $\overline{V} \subset X_\infty \setminus U$, so $\overline{V}$ is compact, since it is a closed subset of a compact set. Thus, X is locally compact at p.

   Now, suppose that X is Hausdorff and locally compact. To show that X_∞ is Hausdorff, we only need to be able to separate ∞ from any point in $p \in X$. Since X is locally compact, there is an open set O so that $p \in O$ and $\overline{O}$ is compact. Then O and $X_\infty \setminus \overline{O}$ are two disjoint open sets in X_∞ containing p and ∞ respectively.

d) If X is compact, then $\{\infty\}$ is an open set in X_∞, since $\{\infty\} = X_\infty \setminus X$. Thus, ∞ is not a limit point of X, and $\overline{X} \neq X_\infty$. Hence, X is not dense.

   If X is not dense in X_∞, then $\overline{X} = X$, since $\infty \notin \overline{X}$. Hence, $\{\infty\}$ is open in X_∞. Thus, X is compact. ∎

Example 43 What is the one-point compactification of the open interval $(0, 1)$? You can define a function $f\colon (0, 1)_\infty \to S^1$ by

$$f(t) = \begin{cases} (\cos(2\pi t), \sin(2\pi t)) & \text{if } 0 < t < 1 \\ (1, 0) & \text{if } t = \infty \end{cases}$$

This f is a one-to-one continuous function from $(0,1)_\infty$ onto the unit circle. By Theorem 55, this is a homeomorphism.