Chapter 1

The Origins of Geometry

1.1 Introduction

In the beginning geometry was a collection of rules for computing lengths, areas, and volumes. Many were crude approximations derived by trial and error. This body of knowledge, developed and used in construction, navigation, and surveying by the Babylonians and Egyptians, was passed to the Greeks. The Greek historian Herodotus (5th century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonians, the Hindu civilization, and the Chinese knew much of what was passed along to the Egyptians.

The Babylonians of 2,000 to 1,600 BC knew much about navigation and astronomy, which required a knowledge of geometry. They also considered the circumference of the circle to be three times the diameter. Of course, this would make $\pi = 3$ — a small problem. This value for π carried along to later times. The Roman architect Vitruvius took $\pi = 3$. Prior to this it seems that the Chinese mathematicians had taken the same value for π. This value for π was sanctified by the ancient Jewish civilization and sanctioned in the scriptures. In I Kings 7:23 we find:

> *He then made the sea of cast metal: it was round in shape, the diameter from rim to rim being ten cubits: it stood five cubits high, and it took a line thirty cubits long to go round it.*
>
> — The New English Bible

Rabbi Nehemiah attempted to change the value of π to 22/7 but was rejected. By 1,800 BC the Egyptians, according to the Rhind papyrus, had the approximation $\pi \approx \left(\frac{16}{9}\right) \approx 3.1604$. To show how far we have progressed, in the 1920's the Indiana legislature passed a law mandating that $\pi = \frac{22}{7}$. (It has since been appealed.) In 1789 Johann Lambert proved that π is an irrational number, and in 1882 F. Lindemann proved that π is transcendental, *i.e.*, it is not the solution to any algebraic equation with rational coefficients.

The ancient knowledge of geometry was passed on to the Greeks. They seemed to be blessed with an inclination toward speculative thinking and the leisure to pursue this inclination. They insisted that geometric statements be established by deductive reasoning rather than trial and error. This began with Thales of Milete. He was familiar with the computations, right or wrong, handed down from Egyptian and Babylonian mathematics. In determining which of the computations were correct, he developed the first logical geometry. This orderly development of theorems by proof was the distinctive characteristic of Greek mathematics and was new.

This new mathematics of Thales was continued over the next two centuries by Pythagoras and his disciples. The Pythagoreans, as a religious sect, believed that the elevation of the soul and union with God were achieved by the study of music and mathematics. Nonetheless, they developed a large body of mathematics by using the deductive method. Their foundation of plane geometry was brought to a conclusion around 400 BC in the *Elements* by the mathematician Hippocrates. This treatise has been lost, but many historians agree that it probably covered most of Books I-IV of Euclid's Elements, which appeared about a century later, circa 300 BC.

> *Euclid was a disciple of the Platonic school. Around 300 BC he produced the definitive treatment of Greek geometry and number theory in his thirteen-volume Elements. In compiling this masterpiece Euclid built on the experience and achievements of his predecessors in preceding centuries: on the Pythagoreans for Books I-IV, VII, and IX, on Archytas for Book VIII, on Eudoxus for Books V, VI, and XII, and on Theætetus for Books X and XIII. So completely did Euclid's work supersede earlier attempts at presenting geometry that few traces remain of these efforts. It's a pity that Euclid's heirs have not been able to collect royalties on his work, for he is the most widely read author in the history of mankind. His approach to geometry has dominated the teaching of the subject for over two thousand years. Moreover, the axiomatic method used by Euclid is the prototype for all of what we now call pure mathematics. It is pure in the sense of pure thought: no physical experiments need be performed to verify that the statements are correct-only the reasoning in the demonstrations need be checked.*

In this treatise, he organized a large body of known mathematics, including discoveries of his own, into the first formal system of mathematics. This formalness was exhibited by the fact that the *Elements* began with an explicit statement of assumptions called axioms or postulates, together with definitions. The other statements — theorems , lemmæ, corollaries — were then shown to follow logically from these axioms and definitions. Books I-IV, VII, and IX of the work dealt primarily with mathematics which we now classify as geometry, and the entire structure is what we now call Euclidean geometry.

Euclidean geometry was certainly conceived by its creators as an idealization of physical geometry. The entities of the mathematical system are concepts, suggested by, or abstracted from, physical experience but differing from physical entities as an idea of an object differs from the object itself. However, a remarkable correlation existed between the two systems. The angle sum of a mathematical triangle was stated to be 180, if one measured the angles of a physical triangle the angle sum did indeed seem to be 180, and so it went for a multitude of other relations. Because of this agreement between theory and practice, it is not surprising that many writers came to think of Euclid's axioms as self evident truths. Centuries later, the philosopher Immanuel Kant even took the position that the human mind is essentially Euclidean and can only conceive of space in Euclidean terms. Thus, almost from its inception, Euclidean geometry had something of the character of dogma.

Euclid based his geometry on five fundamental assumptions:

Postulate I: *For every point P and for every point Q not equal to P there exists a unique line ℓ that passes through P and Q.*

Postulate II: *For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.*

Postulate III: *For every point O and every point A not equal to O there exists a circle with center O and radius OA.*

Postulate IV: *All right angles are congruent to each other.*

Before we study the Fifth Postulate, let me say a few words about his definitions. Euclid's methods are imperfect by modern standards. He attempted to define everything in terms of a more familiar notion, sometimes creating more confusion than he removed. As an example:

A point is that which has no part.

A line is breadthless length. A straight line is a line which lies evenly with the points on itself.

A plane angle is the inclination to one another of two lines which meet. When a straight line set upon a straight line makes adjacent angles equal to one another, each of the equal angles is a right angle.

Euclid did not define length, distance, inclination, or set upon. Once having made the above definitions, Euclid never used them. He used instead the rules of interaction between the defined objects as set forth in his five postulates and other postulates that he implicitly assumed but did not state.

Postulate V: *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

No one seemed to like this Fifth Postulate, possibly not even Euclid himself-he did not use it until Proposition 29. The reason that this statement seems out of place is that the first four postulates seem to follow from experience—try to draw more than one line through 2 different points. The Fifth Postulate is unintuitive. It does come from the study of parallel lines, though. Equivalent to this postulate is :

Playfair's Postulate: *Given a line and a point not on that line, there exists one and only one line through that point parallel to the given line.*

Euclid's Fifth Postulate seemed to be too burdensome. It should follow from the other axioms. Since it is not intuitive, we should be able to prove it as a theorem. We should be able to prove that it is dependent in this Axiom system. If we have a set of axioms A_1, A_2, $\ldots$,A_n for our mathematical system and we can prove that Axiom A_n is derivable, or provable, from the other axioms, then A_n is indeed redundant. In some sense we are looking for a basis for this mathematical system. Unlike vector spaces and linear algebra, there is not a unique number of elements in this basis, for it includes the axioms, definitions, and the rules of logic that you use.

Many people have tried to prove the Fifth Postulate. The first known attempt to prove Euclid V, as it became known, was by Posidionius(1st century BC). He proposed to replaced the definition of parallel lines (those that do not intersect) by defining them as coplanar lines that are everywhere equidistant from one another. It turns out that without Euclid V you cannot prove that such lines exist. It is true that such a statement that parallel lines are equidistant from one another is equivalent to Euclid V.

Ptolemy followed with a proof that used the following assumption:
For every line ℓ and every point P not on ℓ, there exists at most one line m through P such that m is parallel to l.
We will show at a later date that this statement is equivalent to Euclid V, and therefore this did not constitute a proof of Euclid V.

Proclus (410-485A.D.) also attempted to prove Euclid V. His argument used a limiting process. He retained all of Euclid's definitions, all of his assumptions except Euclid V, and hence all of his propositions which did not depend on Euclid V. His plan was (1) to prove on this basis that a line which meets one of two parallels also meets the other, and (2) to deduce Euclid V from this proposition. His handling of step (2) was correctly handled. The argument in step (1) runs substantially as follows. Let g and h be parallel lines and let another line k meet h at P. From Q, a point of k situated between g and h, drop a perpendicular to h. As Q recedes indefinitely far from P, its distance QR from h increases and exceeds any value, however great. In particular, QR will exceed the distance between g and h. For some position of Q, then, QR will equal the distance between g and h. When this occurs, k will meet g.

There are a number of assumptions here which go beyond those found in Euclid. I will mention only the following two:

- the distance from one of two intersecting lines to the other increases beyond all bounds as we recede from their common point,
- the distance between two parallels never exceeds some finite value.

The first of the two assumptions is not a grave error on the part of Proclus, for it can be proved as a theorem on the basis of what he assumed from Euclid. Unfortunately for Proclus, his second assumption is equivalent to Euclid V.

Nasiraddin (1201-1274), John Wallis (1616-1703), Legendre (1752-1833), Wolfgang Bolyai, Girolamo Saccheri (1667-1733), Johann Heinrich Lambert (1728-1777), and many others tried to prove Euclid V, and failed. In these failures there developed a goodly number of substitutes for Euclid V; *i.e.*, statements that were equivalent to the statement of Euclid V. The following is a list of some of these that are more common:

1. *Through a point not on a given line there passes not more than one parallel to the line.*
2. *Two lines that are parallel to the same line are parallel to each other.*
3. *A line that meets one of two parallels also meets the other.*
4. *If two parallels are cut by a transversal, the alternate interior angles are equal.*
5. *There exists a triangle whose angle-sum is two right angles.*
6. *Parallel lines are equidistant from one another.*
7. *There exist two parallel lines whose distance apart never exceeds some finite value.*

8. *Similar triangles exist which are not congruent.*

9. *Through any three non-collinear points there passes a circle.*

10. *Through any point within any angle a line can be drawn which meets both sides of the angle.*

11. *There exists a quadrilateral whose angle-sum is four right angles.*

12. *Any two parallel lines have a common perpendicular.*

It fell to three different mathematicians to independently show that Euclid V is not provable from the other axioms and what is derivable from them. These were Carl Friedrich Gauss, Jnos Bolyai, and Nicolai Ivanovich Lobachevsky. Once these men broke the ice, the pieces of geometry began to fall into place. More was learned about non-Euclidean geometries—hyperbolic and elliptic or doubly elliptic. The elliptic geometry was studied by Riemann, gave rise to riemannian geometry and manifolds, which gave rise to differential geometry which gave rise to relativity theory, *et al.*

This gives us something to anticipate as we learn more about geometry. We will spend our time studying hyperbolic geometry, for it lends itself to better study—not requiring major changes in the axiom system that we have chosen. We may have an opportunity to see that hyperbolic geometry is now lending itself to considerations in the latest research areas of mathematics.