

Chapter 2

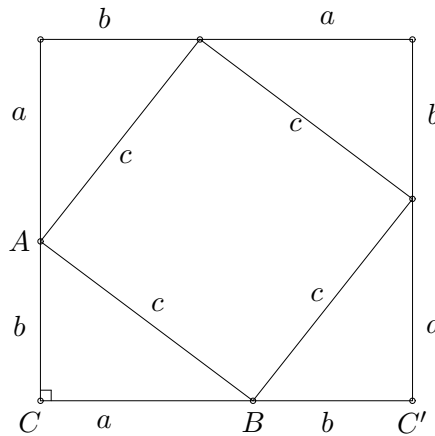
Euclidean Geometry

2.1 The Pythagoreans

Consider possibly the best known theorem in geometry.

Theorem 2.1 (The Pythagorean Theorem) *Suppose a right angle triangle $\triangle ABC$ has a right angle at C , hypotenuse c , and sides a and b . Then*

$$c^2 = a^2 + b^2.$$



PROOF: On the side AB of $\triangle ABC$, construct a square of side c . Draw congruent triangles on each of the other three sides of this square, as in Figure 2.1.

Since the angles at A and B sum to 90° , the angle CBC' is 180° . That means that we have a line. Thus, the resulting figure is a square. The area of the larger square can be calculated in two different ways. First, it is a square of side $a + b$. Second, we add together the area of the square and the four triangles.

$$\begin{aligned}(a + b)^2 &= 4\left(\frac{1}{2}ab\right) + c^2 \\ a^2 + 2ab + b^2 &= 2ab + c^2 \\ a^2 + b^2 &= c^2\end{aligned}$$

as we wanted. ■

What assumptions did we accept in this proof? There were several.

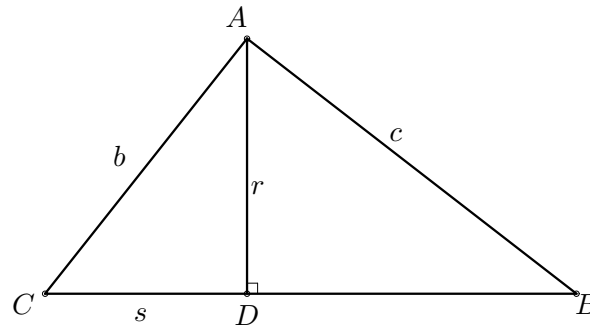
- The area of a square of side s is s^2 .
- The interior angles of a triangle sum to 180° .

Aren't these reasonable assumptions, though? Clearly, any time I draw a square the area will be equal to the square of the side. Is that true? Is it always true that the interior angles of a triangle sum to 180° ? Consider drawing the square on a piece of paper and then lay the paper on a globe. If the globe is big, with respect to the paper, then the square looks pretty much like it does on the flat paper. On the other hand, consider the triangle on the globe that is made from the the Prime Meridian, the Equator and the line of longitude at 90° W. Each of these lines meets the other at a 90° angle. Thus the sum of the interior angles is 270° — much more than 180° . In fact, any triangle drawn on the surface of the globe will have an angle sum more than 180° . Also, the area of the square drawn on the globe will be slightly more than the square of the side. In our *flat* frame of reference the errors are too small to detect.

Theorem 2.2 (The Converse of the Pythagorean Theorem) *Suppose we are in a geometry where the Pythagorean theorem is valid. Suppose that in triangle $\triangle ABC$ we have*

$$a^2 + b^2 = c^2.$$

Then the angle at C is a right angle.



PROOF: As in Figure 2.1 let the perpendicular at A intersect the line BC at the point D . Let $r = |AD|$ and $s = |DC|$. Then by the Pythagorean theorem, $r^2 + s^2 = b^2$ and $r^2 + (a \pm s)^2 = c^2$. The choice of sign depends on whether C is acute or obtuse. Thus, expanding the second equation and substituting the first gives

$$a^2 \pm 2sa + b^2 = c^2.$$

Since $c^2 = a^2 + b^2$, we have that $2sa = 0$. Thus $s = 0$ and $D = C$ making C a right angle. ■

2.2 Euclid's Axioms for Geometry

I mentioned Euclid's Axioms earlier. Now, we want to be more careful in the way that we frame the axioms and make our definitions. This is the basis with which we must work for the rest of the semester. If we do a bad job here, we are stuck with it for a long time.

Since we do not want to have to second guess everything that we prove, we will want to agree on some facts that are absolute and unquestionable. These should be accepted by all and should be easily stated. These will be our axioms. Euclid chose to work with five axioms. (Hilbert in his later work chose to work with 16 axioms.)

Postulate 1: We can draw a unique line segment between any two points.

Postulate 2: Any line segment can be continued indefinitely.

Postulate 3: A circle of any radius and any center can be drawn.

Postulate 4: Any two right angles are congruent.

Postulate 5: Given a line ℓ and a point P not on ℓ , there exists a unique line ℓ_2 through P which does not intersect ℓ .¹

What assumptions have we made here? First of all, we have assumed that a set of points, called the *Euclidean plane* exists. With this assumption comes the concept of length, of lines, of circles, of angular measure, and of congruence. It also assumes that the plane is two-dimensional. All this in five little sentences.

Let's consider what Hilbert does in his choices, and then what Birkhoff chose.

2.2.1 Hilbert's Axioms for Neutral Geometry

GROUP I : Incidence Axioms

I-1: For every point P and for every point Q not equal to P there exists a unique line ℓ that passes through P and Q .

I-2: For every line ℓ there exist at least two distinct points incident with ℓ .

I-3: There exist three distinct points with the property that no line is incident with all three of them.

GROUP II : Betweenness Axioms

B-1: If $A * B * C$, then A , B , and C are three distinct points all lying on the same line, and $C * B * A$.

B-2: Given any two distinct points B and D , there exist points A , C , and E lying on \overleftrightarrow{BD} such that $A * B * D$, $B * C * D$, and $B * D * E$.

B-3: If A , B , and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

B-4: (PLANE SEPARATION AXIOM) For every line ℓ and for any three points A , B , and C not lying on ℓ :

¹This is actually Playfair's postulate. We will give the statement of Euclid later.

- (a) if A and B are on the same side of ℓ and B and C are on the same side of ℓ , then A and C are on the same side of ℓ .
- (b) if A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ , then A and C are on the same side of ℓ .

GROUP III : Congruence Axioms

- C-1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.
- C-2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.
- C-3: If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.
- C-4: Given any $\angle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from a point A' , then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $A'B'$ such that $\angle B'A'C' \cong \angle BAC$.
- C-5: If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.
- C-6: (*SAS*) If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

GROUP IV: Continuity Axioms

ARCHIMEDES' AXIOM: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray \overrightarrow{AB} emanating from A , then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E .

DEDEKIND'S AXIOM: Suppose that the set of all points on a line ℓ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and *vice versa*. Then there is a unique point, O , lying on ℓ such that $P_1 * O * P_2$ if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1, P_2$.

(The following two Principles follow from Dedekind's Axiom, yet are at times more useful.)

CIRCULAR CONTINUITY PRINCIPLE: If a circle γ has one point inside and one point outside another circle γ' , then the two circles intersect in two points.

ELEMENTARY CONTINUITY PRINCIPLE: If one endpoint of a segment is inside a circle and the other outside the circle, then the segment intersects the circle.

Hilbert also used five undefined terms: point, line, incidence, betweenness, and congruence.

2.2.2 Birkhoff's Axioms for Neutral Geometry

The setting for these axioms is the “Absolute (or Neutral) Plane”. It is universal in the sense that all points belong to this plane. It is denoted by A^2 .

AXIOM 1: There exist nonempty subsets of A^2 called “lines,” with the property that each two points belong to *exactly* one line.

AXIOM 2: Corresponding to any two points $A, B \in A^2$ there exists a *unique* number $d(AB) = d(BA) \in \mathbb{R}$, the *distance* from A to B , which is 0 if and only if $A = B$.

AXIOM 3: (*Birkhoff Ruler Axiom*) If k is a line and \mathbb{R} denotes the set of real numbers, there exists a *one-to-one correspondence* ($X \leftrightarrow x$) between the points $X \in k$ and the numbers $x \in \mathbb{R}$ such that

$$d(A, B) = |a - b|$$

where $A \leftrightarrow a$ and $B \leftrightarrow b$.

AXIOM 4: For each line k there are exactly two nonempty *convex* sets R' and R'' satisfying

$$(a) \quad A^2 = R' \cup k \cup R''$$

(b) $R' \cap R'' = \phi$, $R' \cap k = \phi$, and $R'' \cap k = \phi$. That is, they are *pairwise disjoint*.

(c) If $X \in R'$ and $Y \in R''$ then $XY \cap k \neq \phi$.

AXIOM 5: To each angle $\angle ABC$ there exists a unique real number x with $0 \leq x \leq 180$ which is the (*degree*) measure of the angle

$$x = \angle ABC^\circ.$$

AXIOM 6: If $\overrightarrow{BD} \subset \text{Int}(\angle ABC)$, then

$$\angle ABD^\circ + \angle DBC^\circ = \angle ABC^\circ.$$

AXIOM 7: If \overrightarrow{AB} is a ray in the edge, k , of an open half plane $H(k; P)$ then there exist a *one-to-one correspondence* between the open rays in $H(k; P)$ emanating from A and the set of real numbers between 0 and 180 so that if $\overrightarrow{AX} \leftrightarrow x$ then

$$\angle BAX^\circ = x.$$

AXIOM 8: (*SAS*) If a correspondence of two triangles, or a triangle with itself, is such that two sides and the angle between them are respectively congruent to the corresponding two sides and the angle between them, the correspondence is a congruence of triangles.

2.2.3 Return to Euclid's Axioms

The only axioms not listed in either of the two previous lists are parallel axioms. This is because, as we shall see later, there is a choice of parallelism and that will define the geometry.

We need some definitions to work with our choice of Euclid's Axioms.

Definition 2.1 *Distance.* Distance is a real-valued function which assigns to any pair of points in the plane a non-negative real number satisfying the following properties:

$$\begin{aligned} d(P, Q) &= d(Q, P) \\ d(P, Q) &\geq 0, \text{ and there is equality if and only if } P = Q \\ d(P, R) &\leq d(P, Q) + d(Q, R), \text{ the Triangle Inequality.} \end{aligned}$$

We call such a function a metric. We say that the distance from P to Q is $d(P, Q)$. The author of the text will often use $|PQ| = d(P, Q)$.

A line segment is the shortest path between two points. A line is an indefinite continuation of a line segment.

The circle $C_P(r)$ centered at P of radius r is the set of points

$$C_P(r) = \{Q \mid |PQ| = r\}.$$

We will introduce the concept of congruence using the idea of *isometries*.

Definition 2.2 *Isometry.* An isometry of the plane is a map from the plane to itself which preserves distances. That is, f is an isometry if for any two points P and Q in the plane we have

$$d(f(P), f(Q)) = d(P, Q).$$

In your dealings with the Euclidean plane you have run across several isometries: *translations*, *rotations*, and *reflections*. We will formalize these definitions a little later.

Definition 2.3 *Congruence.* Two sets of points (defining a triangle, angle, or some other figure) are congruent if there exists an isometry which maps one set to the other.

This idea of congruence is completed by the following axioms, which guarantee the existence of the isometries we will need.

- Postulate 6. Given any points P and Q , there exists an isometry f so that $f(P) = Q$. (Translations are examples of such.)
- Postulate 7. Given a point P and two points Q and R which are equidistant from P , there exists an isometry which fixes P and maps Q to R . (Rotations and reflections are examples of these.)
- Postulate 8. Given any line ℓ , there exists an isometry which fixes every point in ℓ but fixes no other points in the plane. (A reflection through ℓ is such an example.)

Please note that, for example, Postulate 6 does not guarantee the existence of translations. In fact, depending on how translations are defined, translations do not exist in spherical geometry, but Axiom 6 does hold.

Definition 2.4 *Right Angle.* Two lines ℓ_1 and ℓ_2 intersect at right angles if any two adjacent angles at the point of intersection are congruent. That is, they intersect at right angles if there exists an isometry which sends an angle to one of its adjacent angles.

We do need to deal with the Axioms of Completeness, as did Hilbert and Birkhoff. How do we know that a "geometric line" can be put into a one-to-one correspondence with the set of real numbers and why do we need to know that?

2.3 Triangle Congruence

You will notice that in our list of eight postulates we do not mention any way to determine if two triangles are congruent, other than the definition. We had several methods at our disposal when we studied geometry earlier. What happened to them?

In all honesty, we are looking at geometry in a different way. We are using the group of isometries to determine the geometry. This is the approach that Felix Klein advocated in his Erlangen Programme. We will see that by determining just exactly what the isometries are in a particular situation, we will be able to describe the geometry of the situation.

We are used to having at least three congruence criteria for triangles: side-angle-side (SAS), angle-side-angle (ASA), and side-side-side (SSS). Shouldn't those hold here? Of course, but we will have to prove them from our postulates. We will want to do so without using the parallel postulate so that we know that they will be valid in our *neutral geometry*, or a geometry without a parallel axiom chosen.

If you will remember, the other choices for corresponding parts of triangles did not form congruence criteria when you studied them before: side-side-angle (SSA) and angle-angle-angle (AAA). However, angle-angle-side (AAS) did set up a congruence. Note that if you will remember AAS worked because we appealed to the fact that all triangles summed to 180° , so we could then state that the third angles were congruent and we had reduced this situation to ASA. This proof depends on the Euclidean parallel postulate, so we would want to try to prove this differently, if it is true in neutral geometry.

The proof that we will give depends on a consequence of one of the Continuity Axioms, which we will take up later. Suffice it to say that we will accept the following lemma without proof at this time.²

Lemma 2.1 *Two distinct circles intersect in zero, one, or two points. If there is exactly one point of intersection, then that point lies on the line joining the two centers.*

Theorem 2.3 (SSS) *If the corresponding sides of two triangles $\triangle ABC$ and $\triangle DEF$ have equal lengths, then the two triangles are congruent.*

PROOF: Recall that the definition of congruence requires us to produce an isometry ϕ so that $\phi(A) = D$, $\phi(B) = E$, and $\phi(C) = F$.

First, assume that the triangles are not degenerate (*i.e.*, that each of $\{A, B, C\}$ and $\{D, E, F\}$ form sets of non-collinear points). If they are degenerate, then you should be able to prove this relatively easily from the Triangle Inequality.

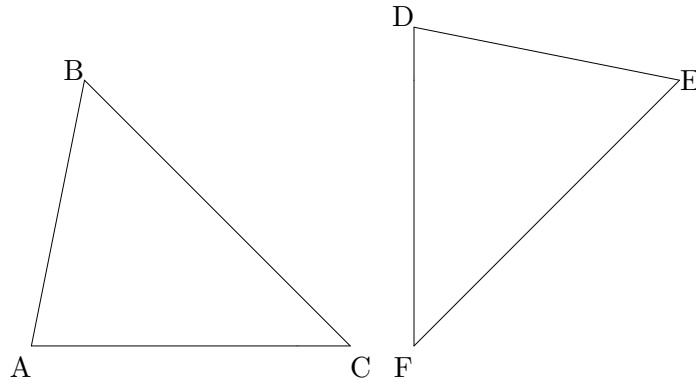
Now, by Axiom 6 there must be an isometry f_1 that sends A to D . Now, f_1 is an isometry and $|AB| = |CD|$, so

$$|Df_1(B)| = |f_1(A)f_1(B)| = |AB| = |DE|,$$

so by Postulate 7, there exists an isometry f_2 such that $f_2(D) = D$ and $f_2(f_1(B)) = E$. Now, if $f_2(f_1(C)) = F$, then we are done because the isometry $f_2 \circ f_1$ is the necessary isometry.

So, assume that $f_2(f_1(C)) \neq F$ and consider the circle centered at D with radius AC and the circle centered at E with radius BC . By Lemma 2.1 these two circles intersect in at most two points. One of these points is F and the other must be $f_2(f_1(C))$. By Postulate

²Its proof does not depend on the result of the theorem we are going to prove.



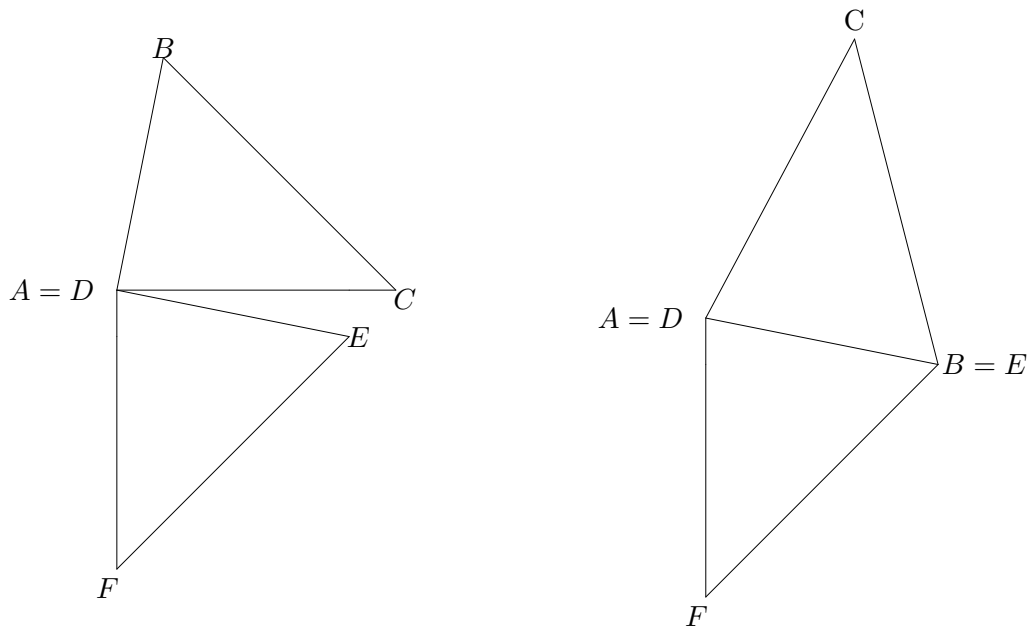
8 there is an isometry f_3 which fixes every point on DE but fixes no other point. Since F is not on DE it must be mapped to another point and that point must be $f_2(f_1(C))$, and *vice versa*. Let $\phi = f_3 \circ f_2 \circ f_1$. Then

$$\begin{aligned}\phi(A) &= D \\ \phi(B) &= E \\ \phi(C) &= F\end{aligned}$$

and the two triangles are congruent, by definition. ■

When one triangle is congruent to another, we will write $\triangle ABC \equiv \triangle A'B'C'$.

Since we indicated earlier that the isometries are going to help us determine the geometry, we need to categorize the isometries. To do this, we need to understand that there is a right hand and a left hand. We must choose which direction is positive, and it is a choice —



it is not something determined *a priori*. This choice is called an *orientation*. More formally, a nondegenerate triangle $\triangle ABC$ is said to be oriented clockwise if the path from A to B to C is oriented clockwise. If a nondegenerate triangle is not oriented clockwise, then we say it is oriented counterclockwise.

Definition 2.5 *Direct Isometry.* An isometry is a direct isometry or a proper isometry if the image of every clockwise triangle is oriented clockwise. An isometry which is not direct is called a improper isometry. Due to the baggage associated with this terminology (and due to tradition) we often call these orientation preserving and orientation reversing isometries.

Definition 2.6 *Translation.* An isometry φ is a translation if φ is an orientation preserving isometry and either φ is the identity or φ has no fixed points.

Definition 2.7 *Rotation.* An isometry φ is a rotation if φ is an orientation preserving isometry and either φ is the identity or there is exactly one point P such that $\varphi(P) = P$. We call P the center of rotation for φ .

Definition 2.8 *Reflection.* An isometry φ is a reflection through the line ℓ if $\varphi(P) = P$ for every $P \in \ell$ and if $\varphi(P) \neq P$ if $P \notin \ell$.

2.4 Euclid's Real Fifth Postulate

Euclid stated his postulate in a *less favorable* form. We believe this for several reasons. First, the statement is:

Postulate 5: *Suppose a line meets two other lines so that the sum of the angles on one side is less than two right angles. Then the two other lines meet at a point on that side.*

This is not a nice, simple statement. Apparently, Euclid did not like the statement because he did not use it until Proposition 29 in his elements. We stated it in the form of Playfair's Postulate. We need to show that these are equivalent.

Theorem 2.4 *Let P be a point not on ℓ , and let Q lie on ℓ so that PQ is perpendicular to ℓ ³ Let ℓ_2 be the line through P which is parallel to ℓ (as guaranteed by Playfair's Postulate). Then ℓ_2 intersects PQ at a right angle.*

PROOF: Assume that ℓ_2 is not perpendicular to PQ . Let ℓ_3 denote the reflection of ℓ_2 through the line PQ . Now, $\ell_3 \neq \ell_2$. Since ℓ is perpendicular to PQ , the reflection of ℓ through PQ is itself. Thus, ℓ_3 cannot intersect ℓ , because if it did then the reflection of the point of intersection would be a point of intersection between ℓ_2 and ℓ , which do not intersect. This gives us more than one line through P which is parallel to ℓ , which contradicts Postulate 5, so ℓ_2 must be perpendicular to PQ . ■

You might ask how we know that this point Q exists. If it does not exist, we would not have a contradiction. Never fear, we have its existence due to the next lemma.

Lemma 2.2 *Let ℓ be a line and P a point not on ℓ . Then there exists a point Q on ℓ so that ℓ is perpendicular to PQ .*

³ Q is called the *foot* of P in ℓ .

PROOF: See the text. ■

Note that the converse of Theorem 2.4 is also true.

Theorem 2.5 *Suppose that ℓ is perpendicular to ℓ_1 and ℓ_2 , then ℓ_1 is parallel to ℓ_2 .*

PROOF: See the text. ■

Corollary 1 *Suppose that ℓ intersects two other lines ℓ_1 and ℓ_2 so that the alternate interior angles are congruent. Then ℓ_1 and ℓ_2 are parallel.*

PROOF: We will prove this much as the text does, but I want you to note that we do not need to appeal to a parallel postulate in order to prove this theorem.

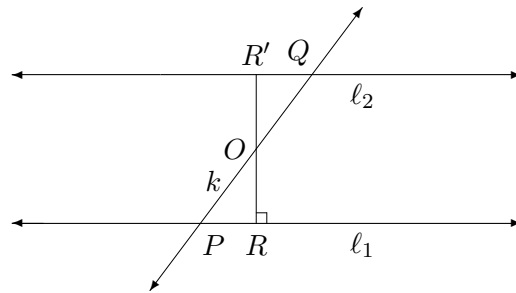


Figure 2.1:

Let k intersect ℓ_1 and ℓ_2 at P and Q , respectively, as in Figure 2.1. Let M be the midpoint of PQ . Drop a perpendicular from O to ℓ_1 and let the foot of the perpendicular be R , thus MR is perpendicular to ℓ_1 . Now, consider the rotation centered at M which sends P to Q . Let the image of R under this rotation be R' . Now R' does not *a priori* lie on the line ℓ_2 . However, we do have that $\triangle MRP \equiv \triangle MR'Q$ since one is the image of the other under an isometry. Now, $\angle MPR \equiv \angle MQR'$, since the alternate interior angles are congruent. Thus, R' must lie on ℓ_2 . Therefore, $\angle PRM \equiv \angle QR'M$ are right angles. Thus, by Theorem 2.4 ℓ_1 and ℓ_2 are parallel. ■

Note that this is not the usual manner in which you encounter this theorem. You usually encounter its converse:

Theorem 2.6 *Suppose that ℓ_1 and ℓ_2 are parallel and that k is a transversal intersecting ℓ_1 and ℓ_2 . Then, alternate interior angles are congruent.*

This statement is equivalent to Euclid's Fifth Postulate, whereas Corollary 1 is true in Neutral Geometry. You must be careful in stating these results. One is true in much more generality than the other, yet both seem so much the same.

Corollary 2 (Euclid's Axiom V) *Suppose a line ℓ meets two other lines ℓ_1 and ℓ_2 so that the sum of the angles on one side is less than two right angles. Then the two other lines meet at a point on that side.*

PROOF: See the text. ■

Theorem 2.7 *The three angles of a triangle sum to two right angles.*

The best proof of this is the one that you give to students of this result.

2.5 The Star Trek Lemma

This is a very common result in Euclidean geometry. It is only true in Euclidean geometry. It is used often and its proof gives us more practice in theorem-proving.

Let A , B , and C be points on a circle centered at O . We will call angle $\angle BAC$ an *inscribed* angle since it is inscribed in a circle. The angular measure of the arc BC is the measure of the central angle $\angle BOC$, where the angle is measured on the same side of O as the arc. We say that $\angle BAC$ *subtends* the arc BC .

Lemma 2.3 (Star Trek Lemma) *The measure of the inscribed angle is half of the angular measure of the arc it subtends.*

PROOF: There are several cases to the proof of the lemma. We will look only at the case where $\angle BAC$ is an acute angle and the center, O , lies in the interior of the angle, as in Figure 2.2.

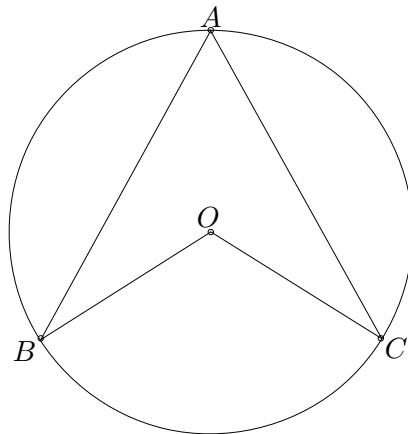


Figure 2.2:

Note that OA , OB and OC are all radii, so we have several isosceles triangles. Extend the segment OA until it meets the circle at a point D . Since $\triangle AOB$ is isosceles, $\angle BAO \equiv \angle OBA$. Also, since the sum of the angles is 180° ,

$$\angle BOD = \angle OBA + \angle BAO = 2\angle BAO.$$

Similarly, $\angle COD = 2\angle CAO$. Thus, adding these together, we have $\angle BOC = 2\angle BAC$. ■

2.6 Similar Triangles

The concept of similar triangles seems so innocuous and so basic, it cannot be related to the Parallel Axiom, can it? It is. It is extremely important in Euclidean geometry. There are numbers of theorems and concepts that rely on similar triangles: slope and trigonometry are just two of these concepts.

Theorem 2.8 *Let B' and C' be on AB and AC , respectively, on the triangle $\triangle ABC$. Then $B'C'$ is parallel to BC if and only if*

$$\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}.$$

Let's defer the proof of this until later — we only want to use it now to see what some of the definitions need to be. Note that since $B'C'$ is parallel to BC , then the corresponding angles of $\triangle ABC$ and $\triangle AB'C'$ are congruent. We call such triangles *similar*.

Definition 2.9 *We say that two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if their corresponding angles are congruent, in which case we write $\triangle ABC \sim \triangle A'B'C'$.*

Lemma 2.4 *If $\triangle ABC \sim \triangle A'B'C'$, then*

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|}.$$

PROOF: Since $\angle BAC \cong \angle B'A'C'$, there is an isometry which sends A' to A and sends B' and C' to points on AB and AC , respectively. Since $\angle ABC \cong \angle A'B'C'$, the line $B'C'$ is parallel to BC , so by Theorem 2.8,

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}.$$

Similarly, by sending B' to B , we can show that

$$\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|}.$$

Combining, we get the desired result. ■

2.7 Power of a Point

Theorem 2.9 *Let P be a point inside a circle Γ . Let QQ' and RR' be two chords which intersect at P . Then*

$$|PQ| \cdot |PQ'| = |PR| \cdot |PR'|.$$

PROOF: By the Star Trek Lemma, $\angle RR'Q \cong \angle RQ'Q$ and $\angle Q'RR' \cong \angle Q'QR'$. Since the angles at P are vertical angles, the triangles $\triangle PQ'R'$ and $\triangle PR'Q$ are similar. Thus,

$$\frac{|PR|}{|PQ|} = \frac{|PQ'|}{|PR'|},$$

and our result follows by cross-multiplication. ■

Theorem 2.10 *Let P be a point outside a circle Γ . Let QQ' and RR' be two chords which intersect at P . Then*

$$|PQ| \cdot |PQ'| = |PR| \cdot |PR'|.$$

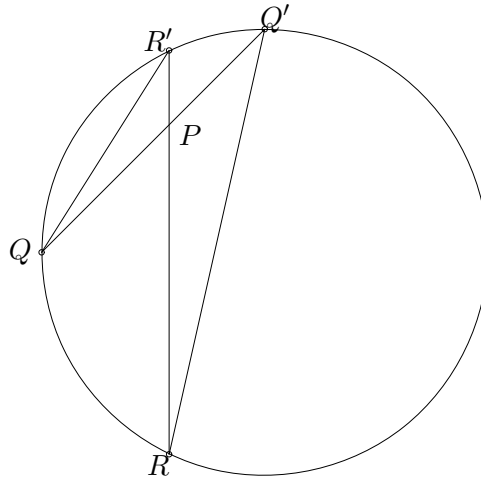


Figure 2.3:

PROOF: This time our setup is slightly different.

However, it is easy to see how to proceed. From Theorem 2.8 we have that $\angle PQ'R \cong \angle PR'Q$. The angle $\angle P$ is shared by the two triangles. Using the fact that the angle sum of a triangle is 180° , then the third angles are equal. Thus, $\triangle PQ'R \cong \triangle PR'Q$. Setting up the appropriate ratios gives us the result. ■

Thus, for any point P and any chord of the circle Γ , QQ' , the product $\Pi(P) = \pm|PQ||PQ'|$ is a constant in absolute value. This is defined to be the *power of a point with respect to a circle*. We choose the sign to be positive if P is outside the circle and negative if P is inside the circle.

Assume that the circle has center O and radius r . Then, choose QQ' to be a diameter of Γ that goes through P . If P is outside Γ , It then follows that

$$|PQ| = |OP| - |OR| = |OP| - r \text{ and } |PQ'| = |OP| + |OR| = |OP| + r.$$

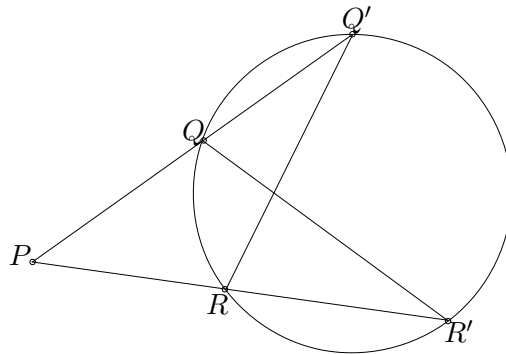


Figure 2.4:

Thus, $\Pi(P) = |PQ||PQ'| = |OP| - r^2$. I leave it to you to check that the same is true if P lies inside Γ .

2.8 Medians and Centroid

In $\triangle ABC$ let A' , B' , and C' be the midpoints of the sides BC , AC , and AB respectively. The line segments AA' , BB' , and CC' are called the *medians* of $\triangle ABC$.

Theorem 2.11 *The three medians of a triangle $\triangle ABC$ intersect at a common point G . Furthermore,*

$$\frac{|AG|}{|A'G|} = \frac{|BG|}{|B'G|} = \frac{|CG|}{|C'G|} = 2.$$

The common point of intersection is called the *centroid* of the triangle $\triangle ABC$.

2.9 Incircle, Excircles, and Law of Cosines

Theorem 2.12 *The angle bisectors of a triangle intersect at a common point I called the incenter, which is the center of the unique circle inscribed in the triangle (called the incircle).*

PROOF: Consider the angle $\angle ABC$ and let D be a point on the angle bisector. Let E and E' be the points on BA and BC , respectively, so that $\angle BED$ and $\angle BE'D$ are right angles. Thus, $\triangle BED \cong \triangle BE'D$ by AAS, since they share BD . Thus, $|DE| = |DE'|$ and the circle centered at D with radius $|DE|$ is tangent to both BA and BC .

Let I be the intersection of the angle bisectors of $\angle ABC$ and $\angle ACB$. The perpendiculars from I to AB and BC are congruent from what we saw above. Likewise, the perpendiculars from I to BC and AC are congruent. Thus, the perpendiculars from I to AB and AC are congruent, so I lies on the angle bisector of $\angle BAC$. (Why?) Thus, the three angle bisectors intersect at a common point. ■

For a triangle $\triangle ABC$ we can define two angle bisectors at each vertex. We have the interior angle bisector at A , about which we just studied. We also have an *exterior angle bisector* at A . Note, that this is NOT the extension of the interior angle bisector to the exterior of the triangle. It is the angle bisector of the angle supplementary to the angle, $\angle BAC$.

We can define three excenters, I_a , I_b , and I_c , as follows. The excenter I_a is the point of intersection of the interior angle bisector of A and the exterior angle bisectors at B and C . It is the center of a circle which is tangent to BC and the extended sides AB and AC , and lies outside $\triangle ABC$. This circle is called an *excircle*.

Let the *inradius* r be the radius of the incircle, and let r_a , r_b , and r_c be the *exradii*. Let $s = \frac{1}{2}(a + b + c)$ be the *semiperimeter* of $\triangle ABC$.

Theorem 2.13 *Let r be the inradius of $\triangle ABC$, and let s be the semiperimeter of $\triangle ABC$. Then*

$$\text{area}(\triangle ABC) = |\triangle ABC| = rs.$$

Theorem 2.14 (Law of Cosines) *For any triangle $\triangle ABC$, we have*

$$c^2 = a^2 + b^2 - 2ab \cos(C).$$

PROOF: Let D be the altitude dropped from A to BC . Then by the Pythagorean Theorem

$$c^2 = |AD|^2 + |DB|^2.$$

Now,

$$\begin{aligned} |AD| &= b \sin(C) \\ |DB| &= |a - b \cos(C)| \end{aligned}$$

Thus,

$$\begin{aligned} c^2 &= b^2 \sin^2(C) + a^2 - 2ab \cos(C) + b^2 \cos^2(C) \\ c^2 &= a^2 + b^2 - 2ab \cos(C) \end{aligned}$$

as we needed. ■

Theorem 2.15 (Heron's Formula) *For any triangle $\triangle ABC$*

$$|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)}.$$

PROOF: Note that

$$|\triangle ABC| = \frac{1}{2}ab \sin(C).$$

By the Law of Cosines,

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

Thus, applying some algebra

$$\begin{aligned} |\triangle ABC| &= \frac{1}{2}ab\sqrt{1 - \cos^2(C)} \\ &= \frac{1}{2}ab\frac{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}{2ab} \\ &= \frac{1}{4}\sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \\ &= \frac{1}{4}\sqrt{((a+b)^2 - c^2)(c^2 - (a-b)^2)} \\ &= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(c-a+b)(c+a-b)} \\ &= \sqrt{\frac{a+b+c}{2} \frac{a+b-c}{2} \frac{-a+b+c}{2} \frac{a-b+c}{2}} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

Heron's formula is named for Heron of Alexandria, who lived sometime between 100 BC and 300 AD. We know that the formula dates back to at least Archimedes (ca. 250 BC). ■

2.10 The Circumcenter and its Spawn

We have seen the *centroid*—center of mass — and the *incenter*. There is yet another center of a triangle. We remember that given any three points there is a unique circle passing through them. How do you find that circle?

Take the perpendicular bisectors of the sides of a triangle formed by the three points. These bisectors meet in a common point, called the *circumcenter*. The radius of the circumcircle is called the circumradius.

Theorem 2.16 *Given a triangle $\triangle ABC$, the perpendicular bisectors of the sides are concurrent. The point is the center of a circle which passes through the vertices of the triangle. The point is called the circumcenter of the triangle.*

PROOF: We must have that two of the perpendicular bisectors intersect. Let p_1 and p_2 denote the perpendicular bisectors of AB and AC respectively. If p_1 is parallel to p_2 , then since AC is perpendicular to p_2 , AC is perpendicular to p_1 . Since AB is perpendicular to p_1 , then AB must be parallel to AC or they coincide. Thus, we would not have a triangle.⁴ Thus, two perpendicular bisectors intersect in a point O . Let M denote the midpoint of AB . Then $\triangle AOM \cong \triangle BOM$, since the angle at M is a right angle, $AM \cong BM$, and $OM \cong OM$. Hence, $AO \cong BO$. Using AC we can also show that $AO \cong CO$. Thus, the triangles $\triangle BON$ and $\triangle CON$ are congruent, where N is the midpoint of BC . Hence, ON is perpendicular to BC and we are done. ■

Theorem 2.17 (Extended Law of Sines) *In triangle $\triangle ABC$*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

PROOF: In $\triangle ABC$, let ON be the perpendicular bisector of BC . Then $\triangle BOC$ is isosceles, $\angle BON \cong \angle CON$ and $BN = CN = a/2$. By the Star Trek Lemma $\angle BOC = 2A$. Thus, $\angle BON = \angle A$. Thus,

$$R \sin A = \frac{a}{2}$$

and

$$2R = \frac{a}{\sin A}.$$

Similarly,

$$2R = \frac{b}{\sin B} = \frac{c}{\sin C},$$

as we needed. ■

2.11 The Euler Line

What happens if the circumcenter, O , coincides with the centroid, G ? That would mean that the medians of the triangle are the perpendicular bisectors as well. This will force the triangle to be an equilateral triangle. What happens if the triangle is not equilateral? Is there any relationship between the circumcenter and the centroid? They will be distinct.

⁴This actually uses a result that is equivalent to Euclid's fifth postulate.

Theorem 2.18 *In an arbitrary triangle, the three altitudes intersect in a common point, called the **orthocenter**.*

There are several ways to prove this. You can do it in a very straightforward manner. However, you will miss a neat result that follows from the following type of proof.

PROOF: If $\triangle ABC$ is equilateral, then the altitudes are the perpendicular bisectors and medians, so the altitudes all meet at $G = O$. Assume that $\triangle ABC$ is not equilateral, so that $O \neq G$.

Let H be the point on the line OG so that $|GH| = 2|OG|$ and the points O , G , and H appear in that order. Let A' be the midpoint of BC , so that $G \in AA'$ and OA' is the perpendicular bisector of BC .

Consider the triangles $\triangle GOA'$ and $\triangle GHA$. Now, $\angle OGA' = \angle HGA$ since they are vertical angles. We have proven that the centroid divides the median in a 2:1 ratio, so $|AG| = 2|GA'|$. We constructed the point H so that $|GH| = 2|OG|$, so the two triangles are similar. Hence, AH is parallel to OA' . If we extend AH to where it intersects BC in a point D , then AD is perpendicular to BC and AD is an altitude of $\triangle ABC$. A similar argument works for the other sides. ■

Theorem 2.19 (The Euler Line) *The circumcenter O , the centroid G , and the orthocenter H are collinear. Furthermore, G lies between O and H and*

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

This line is called the *Euler line*. It was not discovered in any ancient writings and apparently, Leonhard Euler (1707–1783) was the first to discover this result.

2.12 Feuerbach's Circle

The following theorem is not extremely important, but it is "fun". Let A' , B' , and C' be the midpoints of the sides of a triangle $\triangle ABC$. Let D , E , and F be the bases of the altitudes. Let H be the orthocenter, and let A'' , B'' , and C'' be the midpoints of AH , BH , and CH , respectively.

Theorem 2.20 (The Nine Point Circle Theorem) *The nine points A' , B' , C' , A'' , B'' , C'' , D , E , and F all lie on a circle.*

PROOF: B' and C' are midpoints. Therefore, $B'C'$ is parallel to BC .

Consider the triangle $\triangle AHB$. B'' is the midpoint of HB and C' is the midpoint of AB . Thus, $B''C'$ is parallel to AH . Now, remember that AH is perpendicular to BC , so it is perpendicular to $B'C'$. Therefore $B''C'$ is perpendicular to $B'C'$.

Similarly, $B'C''$ is parallel to AH , out of $\triangle AHC$. Also, $B''C''$ is parallel to BC and hence to $B'C'$. Therefore, $B'C'B''C''$ is a rectangle.

Construct the circle with diameter $C'C''$. Since, $\angle C'B''C''$ and $\angle C'B'C''$ are right angles, B' and B'' lie on this circle, and since $|B'B''| = |C'C''|$ we have $B'B''$ is a diameter.

Since CF is an altitude, $\angle C'FC''$ is a right angle, placing F on the circle.

Since $B'B''$ is a diameter and $\angle B'EB''$ is a right angle, E lies on the circle.

Now, make a similar argument to show that $C'A''C''A'$ is a rectangle, so A' and A'' lie on the circle, and $A'A''$ is a diameter, so D lies on the circle. ■

Theorem 2.21 (Feuerbach's Theorem) *The nine point circle of $\triangle ABC$ is tangent to the incircle and the excircles of $\triangle ABC$.*

- Theorem 2.22**
1. *The nine-point circle is the circumcircle of the medial triangle.*
 2. *The nine-point circle has radius one-half that of the circumcircle.*
 3. *The nine-point circle is the circumcircle of the triangle whose vertices are the midpoints of the segments joining $\triangle ABC$'s vertices to the orthocenter.*
 4. *The nine-point circle passes through the points where $\triangle ABC$'s sides are cut by the lines that join $\triangle ABC$'s vertices with its orthocenter.*

2.13 Pedal Triangles and the Simson Line

The Euler line is not unique in the study of triangles. There are other interesting points and lines associated to any triangle.

A *cyclic quadrilateral* is a quadrilateral that can be inscribed in a circle. We proved the following in the homework.

Theorem 2.23 *A convex quadrilateral $ABCD$ is a cyclic quadrilateral if and only if $\angle ABC + \angle CDA = 180^\circ$.*

Let $\triangle ABC$ be an arbitrary triangle and let P be a point either inside or outside the triangle. Let X be the foot of the perpendicular to the extended side BC and through P . Define points Y and Z on the extended sides AC and AB respectively, similarly. The triangle $\triangle XYZ$ is called the *pedal triangle* with respect to the point P and the triangle $\triangle ABC$.

Lemma 2.5 *Let P be a point inside $\triangle ABC$, and let $\triangle XYZ$ be the pedal triangle with respect to P . Then $\angle APB = \angle ACB + \angle XZY$.*

PROOF: Let CP intersect AB at C' . Then write

$$\angle APB = \angle APC' + \angle C'PB.$$

Since $\angle ABC'$ is an exterior angle of $\triangle APC'$, we have that $\angle APC' = \angle PAC + \angle ACP$. Now, $\angle PZA = \angle AYP = 90^\circ$, so they sum to 180° and $AYPZ$ is a cyclic quadrilateral. Thus,

$$\angle PAC = \angle PAY = \angle PZY,$$

which implies $\angle APC' = \angle PZY + \angle ACP$. Similarly, $\angle C'PB = \angle XZP + \angle PCB$. Thus,

$$\begin{aligned} \angle APB &= \angle APC' + \angle C'PB \\ &= (\angle PZY + \angle XZP) + (\angle ACP + \angle PCB) \\ &= \angle XZY + \angle ACB, \end{aligned}$$

as desired. ■

Theorem 2.24 (The Simson Line) *Let Γ be the circumcircle for $\triangle ABC$. Let P be a point on Γ , and let $\triangle XYZ$ be the pedal triangle with respect to P . Then $\triangle XYZ$ is a degenerate triangle, i.e. the points X, Y, Z are collinear. This line is called the Simson line.*

PROOF: Without loss of generality, we may assume P lies on the arc AC . Then $\angle APB = \angle ACB$, since they subtend the same arc. Hence, by Lemma 2.5 $\angle XZY = 0$. That is $\triangle XYZ$ is degenerate. Thus, X, Y , and Z are collinear. ■

2.14 Triangle Centers and Relative Lines

Recall that an excircle of a triangle $\triangle ABC$ is a circle outside the triangle that is tangent to all three of the lines that extend the sides of the triangle. We have three such circles, each tangent to a side and the extensions of the other two sides.

Lemma 2.6 *The lines connecting the point of tangency of each excircle of $\triangle ABC$ to the opposite vertex will intersect in a point, called the Nagel point, N .*

One more point of interest is the center of the incircle for $\triangle ABC$'s medial triangle. This circle is called the Spieker circle and its center is called the Spieker point, S .

Lemma 2.7 *The Nagel segment is a line segment from the incenter, I , to the Nagel point, N , which contains the Spieker point, S , and the centroid, G .*

There is more about this Nagel segment and the Spieker circle.

Lemma 2.8 *For $\triangle ABC$,*

1. *The Spieker circle is the incircle of $\triangle ABC$'s medial triangle.*
2. *The Spieker circle has radius one-half of $\triangle ABC$'s incircle.*
3. *The Spieker circle is the incircle of the triangle whose vertices are the midpoints of the segments that join $\triangle ABC$'s vertices with its Nagel point.*
4. *The Spieker circle is tangent to the sides of $\triangle ABC$'s medial triangle where that triangle's sides are cut by the lines that join $\triangle ABC$'s vertices with its Nagel point.*

Note the similarity to the nine-point circle. In addition, we have the following.

Lemma 2.9 *The Spieker point is the midpoint of the Nagel segment. The centroid is one-third of the way from the incenter to the Nagel point.*

These theorems of concurrence we have considered to this point are related to the concurrence of three lines. Lines are not the only items of interest in geometry. Miquel's Theorem considers the concurrence of sets of three circles associated with a triangle.

Theorem 2.25 (Miquel's Theorem) *If three points are chosen, one on each side of a triangle, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.*

PROOF: Let $\triangle ABC$ be our triangle and let $D, E,$ and F be arbitrary points on the sides of the triangle. Construct the circles determined by pairs of these points and a vertex. Consider two of the circles, C_1 and C_2 , with centers I and J . They must intersect at D , so they must intersect at a second point, call it G . In circle C_2 , we have that the angles $\angle EGD$ and $\angle ECD$ are supplementary. In circle C_1 $\angle FGD$ and $\angle ABD$ are supplementary. Then,

$$\begin{aligned} \angle EGD^\circ + \angle DGF^\circ + \angle EGF^\circ &= 360^\circ \\ (180^\circ - \angle C^\circ) + (180^\circ - \angle B^\circ) + \angle EGF^\circ &= 360^\circ \\ \angle EGF^\circ &= \angle C^\circ + \angle B^\circ = 180^\circ - \angle A^\circ \end{aligned}$$

so that $\angle EGF$ and $\angle EAF$ are supplementary, and hence $E, A, F,$ and G form a cyclic quadrilateral. Thus, all three circles are concurrent. Note that you must modify this proof, slightly, if the Miquel point is outside of the triangle. ■

2.15 Morley's Theorem

Theorem 2.26 (Morley's Theorem) *The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.*

The following proof is due to John Conway.

PROOF: Let the angles $A, B,$ and C measure $3\alpha, 3\beta,$ and 3γ respectively. Let x^+ mean $x + 60^\circ$. Now, we have that $\alpha + \beta + \gamma = 60^\circ$, since $3\alpha + 3\beta + 3\gamma = 180^\circ$. Then there certainly exist seven abstract triangles having the angles:

1	2	3	4	5	6	7
$\alpha^{++}, \beta, \gamma$	$\alpha, \beta^{++}, \gamma$	$\alpha, \beta, \gamma^{++}$	$\alpha, \beta^+, \gamma^+$	$\alpha^+, \beta, \gamma^+$	$\alpha^+, \beta^+, \gamma$	$0^+, 0^+, 0^+$

since in every case the triple of angles adds to 180 degrees. Now these triangles are only determined up to scale, i.e., up to similarity. Determine the scale by saying that certain lines are all to have the same length.

Triangle number 7, with angles $0^+, 0^+, 0^+$, is clearly equilateral, so we can take all its edges to have some fixed length L . Then arrange the edges joining B^+ to C^+ in triangle 4, C^+ to A^+ in triangle 5, and A^+ to B^+ in triangle 6 also to have length L . We will scale the other triangles appropriately.

Then it's easy to see that these all fit together to make up a triangle whose angles are $3A, 3B, 3C$, and which is therefore similar to the original one, so proving Morley's theorem. To see this, you just have to check that any two sides that come together have the same length, and that the angles around any internal vertex add to 360 degrees. The latter is easy, and the former is proved using congruences such as that that takes the vertices A, C^+, B^+ of triangle number 4 to the points A, B^{++}, Y of triangle number 2. ■