Chapter 7

Other Geometries

7.1 The Idea of Parallelism

We have agreed that we would work with a reasonable set of axioms for our geometry. We require that our sets of axioms be *consistent*, *independent* and *complete*.

Definition 7.1 *A set of axioms is said to be consistent if neither the axioms nor the propositions of the system contradict one another.*

Definition 7.2 *A set of axioms is said to be independent if none of the axioms can be derived from any of the other axioms.*

Definition 7.3 *A set of axioms is said to be complete if it is not possible to add a new independent axiom to the system.*

We have looked at Euclid's axioms and have commented on how the first four differ from the Fifth Axiom in that they are direct, concise, and easy to read. The efforts of mathematicians since Euclid's time to show that the Fifth Axiom is dependent on the first four have all met with failure. Geometer's have offered "proofs" of the Fifth Axiom, but each has been found to be flawed. This is one of the times that failure has been helpful, in that they all turn out to be equivalent statements to Euclid's Fifth Axiom.

We are able to show that each of the following statements is logically equivalent to Euclid's Fifth Axiom.

Playfair's Postulate Through a point not on a given line, exactly one parallel may be drawn to the given line.

1. The sum of the angles in a triangle is equal to two right angles.
2. There exists a pair of similar triangles that are not congruent.
3. There exists a pair of lines everywhere equidistant from one another.
4. If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
5. If a line intersects one of two parallel lines, it will intersect the other.

6. Lines parallel to the same line are parallel to one another.

7. Two lines which intersect one another cannot both be parallel to the same line.

Now, we would need to show that each of these is equivalent to Euclid's Fifth Axiom by showing that each implies that Euclid's Fifth Axiom holds and that Euclid's Fifth Axiom implies that each of these is true. You are allowed to use any of the Propositions that Euclid proved without using his Fifth Axiom.

This is an exercise in itself and not where we want to spend our time.

One of the mathematicians who worked on proving that Euclidean geometry was the only geometry was the Italian priest and mathematician Giovanni Girolamo Saccheri.

7.2 Saccheri Quadrilaterals

Saccheri was a Jesuit priest and university professor living from 1667 to 1733. Before he died he published a book entitled *Euclides ab omni nævo vindicatus* (*Euclid Freed of Every Flaw*). It sat unnoticed for over a century and a half until rediscovered by the Italian mathematician Beltrami.

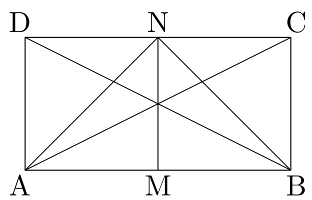
He wished to prove Euclid's Fifth Postulate from the other axioms. To do so he decided to use a *reductio ad absurdum* argument. He assumed the negation of the Parallel Postulate and tried to arrive at a contradiction. He studied a family of quadrilaterals that have come to be called *Saccheri quadrilaterals.* Let S be a convex quadrilateral in which two adjacent angles are right angles. The segment joining these two vertices is called the **base**. The side opposite the base is the **summit** and the other two sides are called the **sides**. If the sides are congruent to one another then this is called a **Saccheri quadrilateral**. The angles containing the summit are called the **summit angles**.

We are able to prove the following:

Theorem 7.1 *In a Saccheri quadrilateral*

i) the summit angles are congruent, and

ii) the line joining the midpoints of the base and the summit—called the **altitude***—is perpendicular to both.*



Now, Saccheri studied the three different possibilities for these summit angles.

Hypothesis of the Acute Angle (HAA)	The summit angles are acute
Hypothesis of the Right Angle (HRA)	The summit angles are right angles
Hypothesis of the Obtuse Angle (HOA)	The summit angles are obtuse

Saccheri intended to show that the first and last could not happen, hence he would have found a proof for Euclid's Fifth Axiom. He was able to show that the Hypothesis of the Obtuse Angle led to a contradiction of what is now know as the Saccheri-Legendre Theorem (see below). He was unable to arrive at a contradiction when he looked at the Hypothesis of the Acute Angle. He gave up rather than accept that there was another geometry available to study.

Theorem 7.2 (Saccheri-Legendre Theorem) *The sum of the degree measures of the three angles in any triangle is* less than or equal to 180°*;*

$$\angle A + \angle B + \angle C \leq 180^\circ.$$

It took another 150 years or so for someone to finally accept the existence of other geometries. Separately Janos Bolyai and Nicolai Lobachevsky discovered new geometries, realized what they had done and published their results. They approached the question of parallelism through Playfair's Postulate. Again, they had to consider three possibilities — and each one leads to a different geometry.

Hyperbolic Axiom	Through a point not on a given line, more than one parallel may be drawn to the given line.
Euclidean Axiom	Through a point not on a given line, exactly one parallel may be drawn to the given line.
Elliptic Axiom	Through a point not on a given line, no parallels may be drawn to the given line.

The geometry that they studied was the first published logically consistent alternative to Euclid. Their discovery is known as hyperbolic (or Lobachevskian) geometry and is characterized by the following axioms:

- Given any two points, exactly one line may be drawn containing these points.
- Given any line, a segment of any length may be determined on the line.
- Given any point, a circle of any radius may be drawn.
- All right angles are congruent.
- Through a point not on a given line, at least two parallel lines may be drawn to the given line.

7.3 Poincaré's Disk Model for Hyperbolic Geometry

If we adopt the *Hyperbolic Axiom* then there are certain ramifications:

1. The sum of the angles in a triangle is less than two right angles.
2. All similar triangles that are congruent, *i.e.* AAA is a congruence criterion.
3. There are no lines everywhere equidistant from one another.
4. If three angles of a quadrilateral are right angles, then the fourth angle is less than a right angle.
5. If a line intersects one of two parallel lines, it may not intersect the other.
6. Lines parallel to the same line need not be parallel to one another.

7. Two lines which intersect one another may both be parallel to the same line.

How can we see this? It cannot be by just looking at the Euclidean plane in a slightly different way. We would like a model with which we could study the hyperbolic plane. If it is to be a Euclidean object that we use to study the hyperbolic plane, H^2, then we must have to make some major changes in our concept of point, line, and distance.

We need a model to see what H^2 looks like. We know that it will not be too easy, but we do not want some extremely difficult model to construct. We will work with a small subset of the plane, but give it a different way of measuring distance.

In order to give a model for H^2, we need to determine the set of points, then determine what lines are and how to measure distance. For Poincaré's Disk Model we take the set of points that lie inside the unit circle, *i.e.*, the set

$$H^2 = \{(x, y) \mid x^2 + y^2 < 1\}.$$

Note that points on the circle itself are NOT in the hyperbolic plane. However they do play an important part in determining our model. Euclidean points on the circle itself are called *ideal points*, *omega points*, *vanishing points*, or *points at infinity.*

A *unit* circle is any circle in E^2[1] is a circle with radius one.

Definition 7.4 *Given a unit circle Σ in the Euclidean plane, points of the hyperbolic plane are the points in the interior of Σ. Points on this unit circle are called omega points (Ω) of the hyperbolic plane.*

If we take Σ to be the unit circle centered at the origin, then we would think of the hyperbolic plane as $H^2 = \{(x, y) \mid x^2 + y^2 < 1\}$ and the omega points are the points $\Omega = \{(x, y) \mid x^2+y^2 = 1\}$. The points in the Euclidean plane satisfying $\{(x, y) \mid x^2+y^2 > 1\}$ are called *ultraideal points.*

We now have what our points will be. We see that we are going to have to modify our concept of line in order to have the Hyperbolic Axiom to hold.

Definition 7.5 *Given a unit circle Σ in the Euclidean plane, lines of the hyperbolic plane are arcs of circles drawn orthogonal*[2] *to Σ and located in the interior of Σ.*

7.3.1 Construction of Lines

This sounds nice, but how do you draw them?

1. Start with a circle Γ centered at O and consider the ray $\overrightarrow{OA}$.

2. Construct the line perpendicular to $\overrightarrow{OA}$ at A.

3. Choose a point P on this perpendicular line for the center of the second circle and make PA the radius of a circle centered at P.

4. Label the second point of intersection with circle Γ B. Then the arc AB represents a line in this model.

[1] the Euclidean plane
[2] Circles are orthogonal to one another when their radii at the points of intersection are perpendicular.

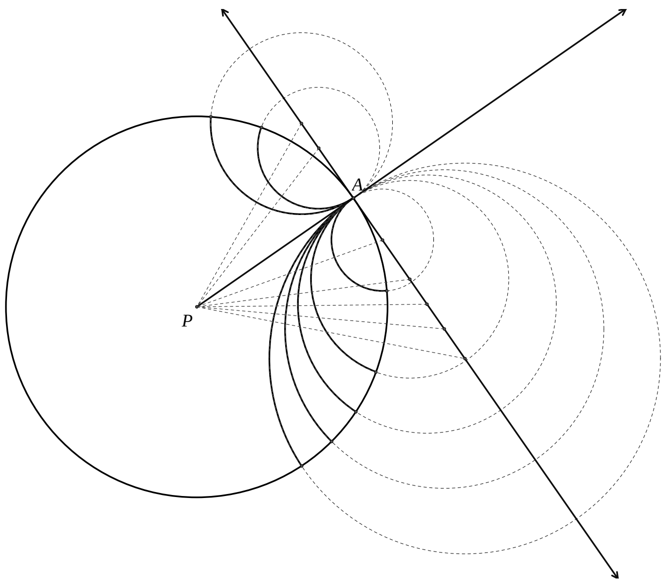


Figure 7.2: Poincaré lines through A

Now, how do you construct these lines in different circumstances? There are three cases we need to consider.
Case I: $A, B \in \Gamma$
Case II: $A \in \Gamma$ and B lies inside Γ
Case III: A and B both lie inside Γ.

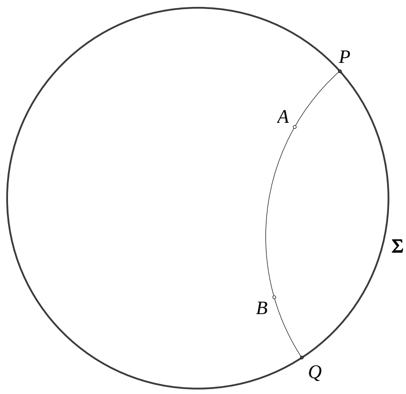


Figure 7.1: Poincaré line

Case I: Construct rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ where P is the center of the circle Γ. Construct the lines perpendicular to $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at A and B respectively. Let Q be the point of intersection of those two lines. The circle Ω centered at Q with radius QA intersects Γ at A and B. The line between A and B is the arc of Ω that lies inside Γ.

Note that this arc is clearly orthogonal to Γ by its construction.

Case II: Construct rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ where P is the center of the circle Γ. Construct the line perpendicular to $\overrightarrow{PA}$ at A. Draw segment AB and construct its perpendicular bisector. Let Q be the point of intersection of this line and the tangent line to Γ at A. The circle Ω centered at Q with radius QA contains A and B. The line containing A and B is the arc of Ω that lies inside Γ.

This arc, as constructed is orthogonal to Γ at A. We want to see that it is orthogonal at the other point of intersection with the circle. Let that point of intersection be X. Then, $X \in \Gamma$ means that $PA \cong PX$. Since X lies on our second circle it follows that $QX \cong QA$. Since $PQ \cong PQ$, we have that $\triangle PAQ \cong \triangle PXQ$, which means that $\angle PXQ$ is a right angle, as we wanted to show.

Case III: Construct the ray $\overrightarrow{PA}$ and then construct the line perpendicular to $\overrightarrow{PA}$ at A. This intersects Γ in points X and Y. Construct the tangents to Γ at X and at Y. These tangent lines intersect at a point C. The circle Ω centered at Q is the circle passing through A, B, and C. The line containing A and B is the arc of Ω that lies inside Γ.

From our construction, we have that $\triangle PXC \sim \triangle PAX$ and it follows that $|PA||PC| = |PX|^2 = r^2$. Now, Q lies on the perpendicular bisectors of AC and AB as Ω is the circumcircle for $\triangle ABC$. There is a point T on the circle Ω so that the tangent line to Ω at T passes through P.

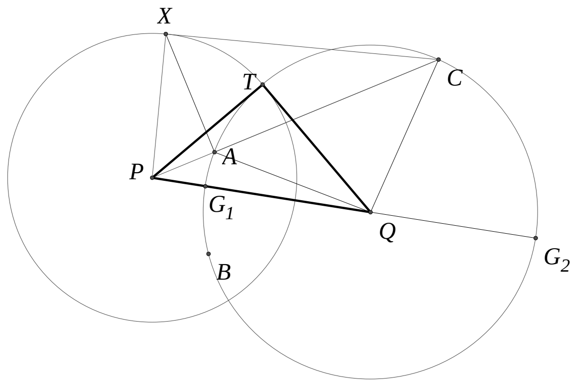


Figure 7.3: Poincaré line in Case III

Construct the line through P and Q which intersects the circle in two points G_1 and G_2 so that G_1 lies between P and Q. Now,

$$\begin{aligned}
|PT|^2 &= |PQ|^2 - |QT|^2 \\
&= (|PQ| - |QT|)\,(|PQ| + |QT|) \\
&= (|PQ| - |QG_1|)\,(|PQ| + |QG_2|) \\
&= |PG_1||PG_2| \text{ which by Theorem 5.3,} \\
&= |PA||PC| = r^2
\end{aligned}$$

Therefore, T lies on the circle Γ and Γ and Ω are orthogonal at that point. A similar argument shows that they are orthogonal at the other point of intersection.

7.3.2 Distance

Now, this Euclidean area inside the unit circle must represent the infinite hyperbolic plane. This means that our standard distance formula will not work. We introduce the distance metric by

$$d\rho = \frac{2dr}{1-r^2}$$

where ρ represents the hyperbolic distance and r is the Euclidean distance from the center of the circle. Note that $d\rho \to \infty$ as $r \to 1$. This means that lines are going to have infinite extent.

The relationship between the Euclidean distance of a point from the center of the circle and the hyperbolic distance is:

$$\rho = \int_0^r \frac{2du}{1-u^2} = \ln\left(\frac{1+r}{1-r}\right) = 2\tanh^{-1} r,$$

or $r = \tanh \dfrac{\rho}{2}$.

Now, we can use this to define the distance between two points on a Poincaré line. Given two hyperbolic points A and B, let the Poincaré line intersect the circle in the omega points P and Q. Let

$$(AB, PQ) = \frac{AP/AQ}{BP/BQ} = \frac{AP \cdot BQ}{AQ \cdot BP},$$

denote the *cross ratio* of A and B with respect to P and Q, where AP denotes the the Euclidean arclength. Define the hyperbolic distance from A to B to be

$$d(A, B) = \log |AB, PQ|.$$

Theorem 7.3 *If a point A in the interior of Σ is located at a Euclidean distance $r < 1$ from the center O, its hyperbolic distance from the center is given by*

$$d(A, O) = \log \frac{1 + r}{1 - r}.$$

Theorem 7.4 *The hyperbolic distance from any point in the interior of Σ to the circle itself is infinite.*

7.3.3 Parallel Lines

It is easy to see that the Hyperbolic Axiom works in this model. Given a line $\overleftrightarrow{AB}$ and a point $D \notin \overleftrightarrow{AB}$, then we can draw at least two lines through D that do not intersect $\overleftrightarrow{AB}$.

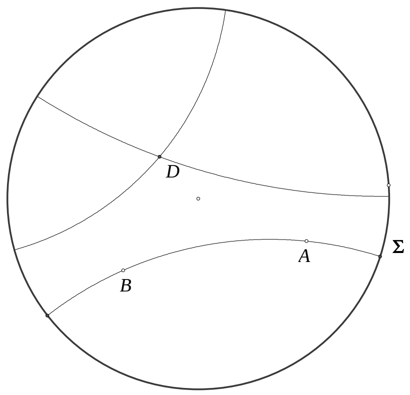


Figure 7.4: Multiple parallels through A

Call these two lines through D lines ℓ_1 and ℓ_2. Notice now how two of our results do not hold, as we remarked earlier. We have that $\overleftrightarrow{AB}$ and ℓ_1 and $\overleftrightarrow{AB}$ and ℓ_2 are parallel, ***but*** ℓ_1 and ℓ_2 are not parallel. Note also that ℓ_2 intersects one of a pair of parallel lines (ℓ_1), but does not intersect the other parallel line ($\overleftrightarrow{AB}$).

The hyperbolic plane has two types of parallel lines. The definition that we will give here will depend explicitly on the model that we have chosen, but we will make a more general definition later. Consider the hyperbolic line $\overleftrightarrow{AB}$ which intersects the circle Σ in the ideal points Λ and Ω. Take a point $D \notin \overleftrightarrow{AB}$. Construct the line through Λ and D. Since this line does not intersect the line $\overleftrightarrow{AB}$ inside the circle, these two hyperbolic lines are parallel. However, they seem to be approaching one another as we go "to infinity". In some sense this is true, as we shall see later. Since there are two "ends" of the Poincaré line $\overset{\frown}{AB}$, there are two of these lines. The line $\overset{\frown}{AB}$ and $\overset{\frown}{D\Lambda}$ are said to be *limiting parallel*.[3] The defining property is as follows.

Definition 7.6 *Let $P \in \overset{\frown}{AB}$. Consider the collection of lines $\overset{\frown}{DP}$ as P goes to Ω or Λ. The first line through D in this collection that does not intersect $\overset{\frown}{AB}$ in H^2 is the limiting parallel line to $\overset{\frown}{AB}$ in that direction.*

[3] These are also called *sensed-parallel*, *ultraparallel*, or *horoparallel*.

Drop a perpendicular from D to $\overset{\frown}{AB}$ and label this point of intersection M. Angles $\angle \Lambda DM$ and $\angle \Omega DM$ are called *angles of parallelism.*

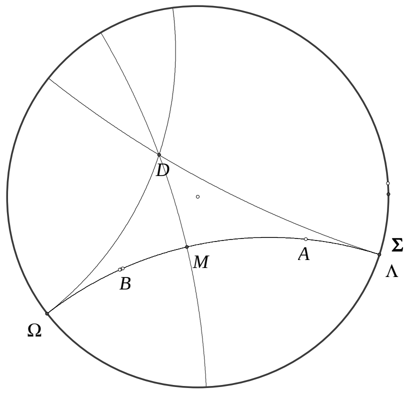


Figure 7.5: Limiting Parallel Poincaré Lines

Theorem 7.5 *The angles of parallelism associated with a given line and point are congruent.*

PROOF: Assume not, *i.e.*, assume $\angle \Lambda DM \neq \angle \Omega DM$. Then one angle is greater than the other. Without loss of generality, we may assume that $\angle \Lambda DM < \angle \Omega DM$. Then there is a point E in the interior of $\angle \Omega DM$ such that $\angle \Lambda DM = \angle EDM$. The line $\overset{\frown}{ED}$ must intersect $\overset{\frown}{AB}$ since $\overset{\frown}{D\Omega}$ is the limiting parallel line to $\overset{\frown}{AB}$ in that direction. Let the point of intersection be F. Choose G on $\overset{\frown}{AB}$ on the opposite side of $\overset{\frown}{DM}$ from F so that $FM = GM$. Then $\triangle GMD \cong \triangle FMD$. This implies that $\angle GDM = \angle FDM = \angle \Lambda DM$. This means that $\overset{\frown}{D\Omega}$ intersects $\overset{\frown}{AB}$ at G. This contradicts the condition that $\overset{\frown}{D\Omega}$ is limiting parallel to $\overset{\frown}{AB}$. Thus, the angles of parallelism are congruent. ∎

Theorem 7.6 *The angles of parallelism associated with a given line and point are acute.*

PROOF: Assume not, *i.e.*, assume that $\angle MD\Omega > 90°$. Then there is a point E interior to $\angle MD\Omega$ so that $\angle MDE = 90°$. Then, since $\overset{\frown}{DE}$ and $\overset{\frown}{AB}$ are perpendicular to the same line, they are parallel. Thus, $\overset{\frown}{DE}$ does not intersect $\overset{\frown}{AB}$ which contradicts the condition that $D\Omega$ is the limiting parallel line.

If the angle of parallelism is 90° then we can show that we have Euclidean geometry. Thus, in H^2 the angle of parallelism is acute. ∎

Theorem 7.7 (Lobachevskii's Theorem) *Given a point P at a hyperbolic distance d from a hyperbolic line $\overset{\frown}{AB}$ (i.e., $d(P, M) = d$), the angle of parallelism, θ, associated with the line and the point satisfies*

$$e^{-d} = \tan\left(\frac{\theta}{2}\right).$$

Note then that

$$\lim_{d \to 0} \theta = \frac{\pi}{2} \text{ and } \lim_{d \to \infty} \theta = 0.$$

PROOF: The proof of this is interesting in that we play one geometry off of the other in order to arrive at our conclusion.

We are given a line $\overset{\frown}{AB}$ and a point P not on the line. Construct the line through P which is perpendicular to $\overset{\frown}{AB}$. Call the point of intersection R as in Figure 7.6. Then we have that $d = d(P, R)$. We can translate P to the center of the unit circle and translate our line to a line so that our line perpendicular to AB is a radius of Σ as we have done in Figure 7.7. Construct the radii from P to the ideal points A and B and construct the lines tangent to Σ at these points. These tangent lines intersect at a point Q which lies on $\overrightarrow{PR}$. Now, since we have moved our problem to the center of the circle, we can use our previous result to see that if r is the Euclidean distance from P to R, then we have

$$d = \log \frac{1 + r}{1 - r},$$

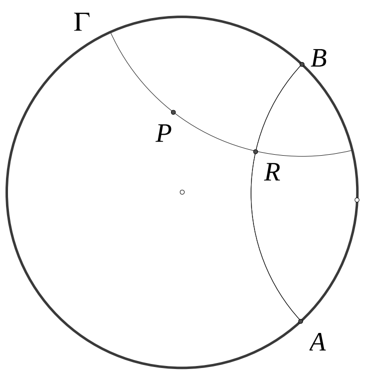


Figure 7.6:

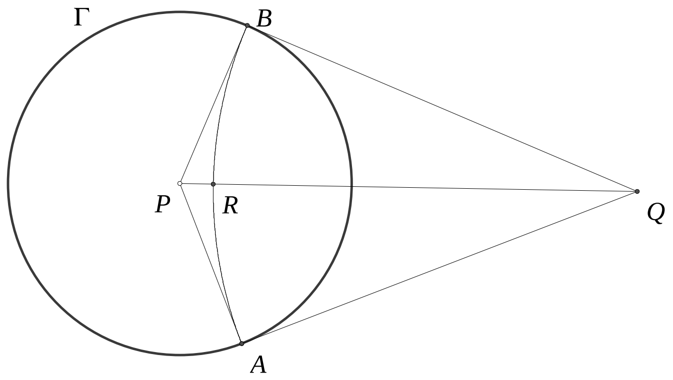


Figure 7.7:

or rewriting this we have

$$e^d = \frac{1+r}{1-r} \text{ or } e^{-d} = \frac{1-r}{1+r}.$$

Now, we are talking about Euclidean distances (with r) and using our Euclidean right triangles with radius 1 we have that:

$$r = QP - QR = QP - QA = \sec\angle QPA - \tan\angle QPA = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}.$$

Now, algebra leads us to:

$$\begin{aligned}
e^{-d} &= \frac{1-r}{1+r} \\
&= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} \\
&= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1}\frac{\cos\theta + \sin\theta + 1}{\cos\theta + \sin\theta + 1} \\
&= \frac{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta - 1}{\cos^2\theta + 2\cos\theta - \sin^2\theta + 1} \\
&= \frac{2\sin\theta\cos\theta}{2\cos^2\theta + 2\cos\theta} = \frac{\sin\theta}{1+\cos\theta} \\
&= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right) + 1} \\
&= \tan\left(\frac{\theta}{2}\right)
\end{aligned}$$

$\blacksquare$

7.3.4 Hyperbolic Circles

Now, if we have a center of a circle that is not at the center P of the unit circle Σ, we know that the hyperbolic distance in one direction looks skewed with respect to the Euclidean distance. That would lead us to expect that a circle in this model might take on an elliptic or oval shape. We will prove later that this is not the case. In fact, hyperbolic circles embedded in Euclidean space retain their circular appearance — their centers are offset!

Theorem 7.8 *Given a hyperbolic circle with radius R, the circumference C of the circle is given by $C = 2\pi \sinh(R)$.*

7.3.5 Similarities with Euclidean Geometry

Because we only changed the Fifth Axiom and not the first four, everything that holds in Neutral Geometry (or geometry without a parallel postulate) also holds in the Hyperbolic Plane. Other things may also hold, but may require a different proof!! Examples of theorems that are still true are:

Theorem 7.9 (Pasch's Theorem) *If $\triangle ABC$ is any triangle and ℓ is any line intersecting side AB in a point between A and B, then ℓ also intersects either side AC or side BC. If $C \notin \ell$, then ℓ does not intersect both AC and BC.*

Theorem 7.10 (Crossbar Theorem) *If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects the segment BC.*

Theorem 7.11 *Supplements of congruent angles are congruent.*

Theorem 7.12 *Vertical angles are congruent to each other.*

Theorem 7.13 *An angle congruent to a right angle is a right angle.*

Theorem 7.14 *For every line ℓ and every point P there exists a line through P perpendicular to ℓ.*

Theorem 7.15 (ASA) *Given $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $AC \cong DF$. Then $\triangle ABC \cong \triangle DEF$.*

Theorem 7.16 (SSS) *Given triangles $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $AC \cong DF$, and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 7.17 (Alternate Interior Angles Theorem) *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are non-intersecting.*

Theorem 7.18 *If m and n are distinct lines both perpendicular to the line ℓ, then m and n are non-intersecting.*

Theorem 7.19 *If P is a point not on ℓ, then the perpendicular dropped from P to ℓ is unique.*

Theorem 7.20 *If ℓ is any line and P is any point not on ℓ, there exists at least one line m through P which does not intersect ℓ.*

Theorem 7.21 (Exterior Angle Theorem) *An exterior angle of a triangle is greater than either remote interior angle.*

Theorem 7.22 (SAA Congruence) *In triangles $\triangle ABC$ and $\triangle DEF$ given that $AC \cong DF$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 7.23 *Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.*

Theorem 7.24 *Every segment has a unique midpoint.*

Theorem 7.25 *Every angle has a unique bisector.*

Theorem 7.26 *Every segment has a unique perpendicular bisector.*

Theorem 7.27 *In a triangle the greater angle lies opposite the greater side and the greater side lies opposite the greater angle; i.e.,* $AB > BC$ *if and only if* $\angle C > \angle A$.

Theorem 7.28 *Given* $\triangle ABC$ *and* $\triangle A'B'C'$, *if* $AB \cong A'B'$ *and* $BC \cong B'C'$, *then* $\angle B < \angle B'$ *if and only if* $AC < A'C'$.