Chapter 12

Hyperbolic Analytic Geometry

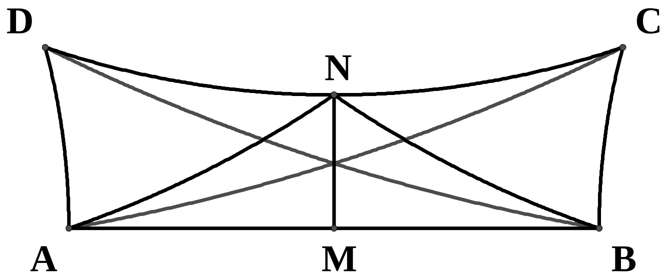
12.1 Saccheri Quadrilaterals

Recall the results on *Saccheri quadrilaterals* from Chapter 4. Let S be a convex quadrilateral in which two adjacent angles are right angles. The segment joining these two vertices is called the **base**. The side opposite the base is the **summit** and the other two sides are called the **sides**. If the sides are congruent to one another then this is called a **Saccheri quadrilateral**. The angles containing the summit are called the **summit angles**.

Theorem 12.1 *In a Saccheri quadrilateral*

i) the summit angles are congruent, and

ii) the line joining the midpoints of the base and the summit—called the ***altitude****—is perpendicular to both.*



Theorem 12.2 *In a Saccheri quadrilateral the summit angles are acute.*

Recall that a convex quadrilateral three of whose angles are right angles is called a **Lambert quadrilateral**.

Theorem 12.3 *The fourth angle of a Lambert quadrilateral is acute.*

Theorem 12.4 *The side adjacent to the acute angle of a Lambert quadrilateral is greater than its opposite side.*

Theorem 12.5 *In a Saccheri quadrilateral the summit is greater than the base and the sides are greater than the altitude.*

12.2 More on Quadrilaterals

Now we need to consider a Saccheri quadrilateral which has base b, sides each with length a, and summit with length c. We showed that $c > a$, but we would like to know

- How much bigger?
- How are the relative sizes related to the lengths of the sides?

Theorem 12.6 *For a Saccheri quadrilateral*

$$\sinh\frac{c}{2} = (\cosh a)\cdot(\sinh\frac{b}{2}).$$

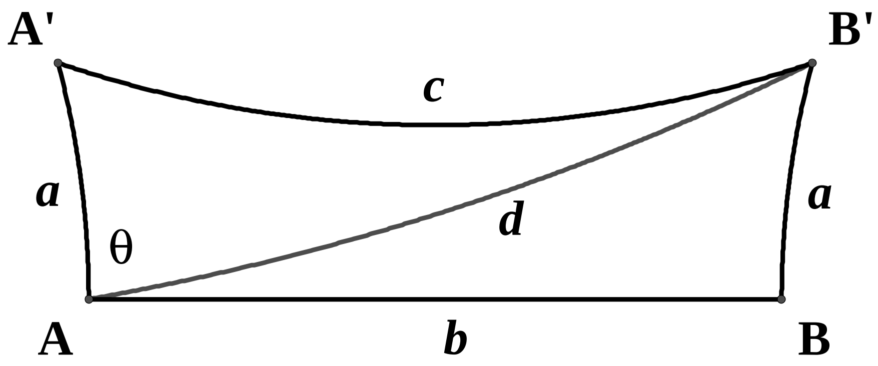


Figure 12.1: Saccheri Quadrilateral

Proof: Compare Figure 12.1. Applying the Hyperbolic Law of Cosines from Theorem 11.2, we have

$$\cosh c = \cosh a\cosh d - \sinh a\sinh d\cos\theta. \tag{12.1}$$

From Theorem 11.1 we know that

$$\cos(\theta) = \sin(\frac{\pi}{2}-\theta) = \frac{\sinh a}{\sinh d}$$
$$\cosh d = \cosh a\cosh b$$

Using these in Equation 12.1 we eliminate the variable d and have

$$\begin{aligned}\cosh c &= \cosh^2 a\cosh b - \sinh^2 a\\ &= \cosh^2 a(\cosh b - 1) + 1\end{aligned}$$

Now, we need to apply the identity

$$2\sinh^2(\frac{x}{2}) = \cosh x - 1,$$

and we have the formula. ∎

Corollary 5 *Given a Lambert quadrilateral, if c is the length of a side adjacent to the acute angle, a is the length of the other side adjacent to the acute angle, and b is the length of the opposite side, then*

$$\sinh c = \cosh a \sinh b.$$

Two segments are said to be *complementary segments* if their lengths x and x^* are related by the equation

$$\Pi(x) + \Pi(x^*) = \frac{\pi}{2}.$$

The geometric meaning of this equation is shown in the following figure, Figure 12.2. These lengths then are complementary if the angles of parallelism associated to the segments are complementary angles. This is then an "ideal Lambert quadrilateral" with the fourth vertex an ideal point Ω.

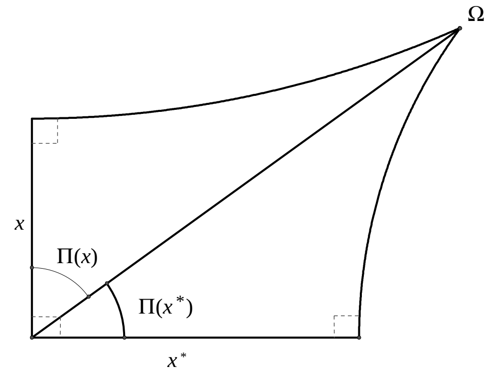


Figure 12.2: Complementary Segments

If we apply the earlier formulas for the angle of parallelism to these segments, we get

$$\begin{aligned} \sinh x^* &= \operatorname{csch} x \\ \cosh x^* &= \coth x \\ \tanh x^* &= \operatorname{sech} x \\ \tanh \frac{x^*}{2} &= e^{-x}. \end{aligned}$$

Theorem 12.7 (Engel's Theorem) *There is a right triangle with sides and angles as shown in Figure 12.3 if and only if there is a Lambert quadrilateral with sides as shown is Figure 12.3. Note that PQ is a complementary segment to the segment whose angle of parallelism is $\angle A$.*

12.3 Coordinate Geometry in the Hyperbolic Plane

In the hyperbolic plane choose a point O for the origin and choose two perpendicular lines through O—OX and OY. In our models—both the Klein and Poincaré—we will use the

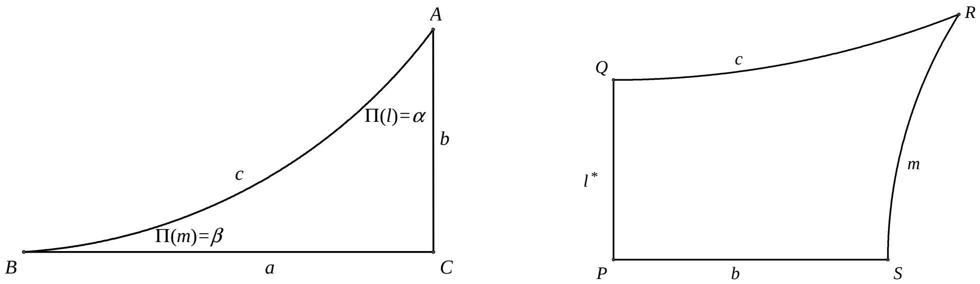


Figure 12.3: Engel's Theorem

Euclidean center of our defining circle for this point O. We need to fix coordinate systems on each of these two perpendicular lines. By this we need to choose a positive and a negative direction on each line and a unit segment for each. There are other coordinate systems that can be used, but this is standard. We will call these the u-axis and the v-axis. For any point $P \in \mathscr{H}^2$ let U and V be the feet of P on these axes, and let u and v be the respective coordinates of U and V. Then the quadrilateral $\square UOVP$ is a Lambert quadrilateral. If we label the length of UP as w and that of VP as z, then by the Corollary to Theorem 12.6 we have

$$\tanh w = \tanh v \cdot \cosh u$$
$$\tanh z = \tanh u \cdot \cosh v$$

Let $r = d_h(OP)$ be the hyperbolic distance from O to P and let θ be a real number so that $-\pi < \theta < \pi$. Then

$$\tanh u = \cos\theta \cdot \tanh r$$
$$\tanh v = \sin\theta \cdot \tanh r.$$

We also set

$$x = \tanh u, \quad y = \tanh v$$
$$T = \cosh u \cosh w, \; X = xT, \; Y = yT.$$

The ordered pair $\{OX, OY\}$ is called a **frame** with **axes** OX and OY. With respect to this frame, we say the point P has

- **axial coordinates** (u, v),
- **polar coordinates** (r, θ),
- **Lobachevsky coordinates** (u, w),
- **Beltrami coordinates** (x, y),
- **Weierstrass coordinates** (T, X, Y).

If a point has Beltrami coordinates (x, y) and $t = 1 + \sqrt{1 - x^2 - y^2}$, put

$$p = x/t \qquad q = y/t,$$

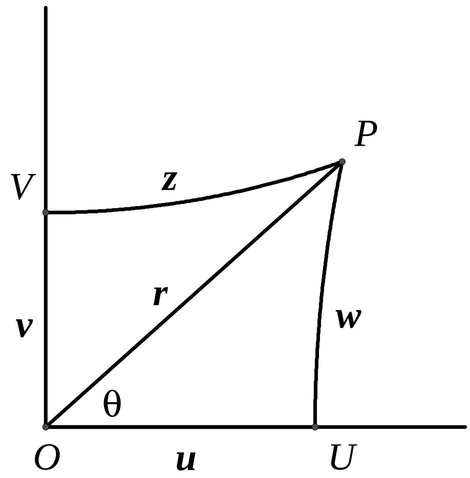


Figure 12.4: Coordinates in Poincaré Plane

then (p, q) are the **Poincaré coordinates** of the point.

In Figure 12.5 we have:

$$\begin{aligned} u &= 0.78 \\ v &= 0.51 \\ w &= 0.72 \\ z &= 0.94 \\ r &= 1.10 \\ \theta &= 35.67^\circ = 0.622\text{radians} \end{aligned}$$

From which it follows that

$$\begin{aligned} x &= \tanh u = 0.653, & y &= \tanh v = 0.470 \\ T &= \cosh u \cosh w = 1.677 & X &= xT = 1.095 \\ Y &= yT = 0.788 & t &= 1 + \sqrt{1 - x^2 - y^2} = 1.594 \\ p &= x/t = 0.409 & q &= y/t = 0.295. \end{aligned}$$

Thus the coordinates for P are:

- axial coordinates $(u, v) = (0.78, 0.51)$,
- polar coordinates $(r, \theta) = (1.10, 0.622)$,
- Lobachevsky coordinates $(u, w) = (0.78, 0.72)$,
- Beltrami coordinates $(x, y) = (0.653, 0.470)$,
- Weierstrass coordinates $(T, X, Y) = (1.677, 1.095, 0.788)$.
- Poincaré coordinates $(p, q) = (0.409, 0.295)$

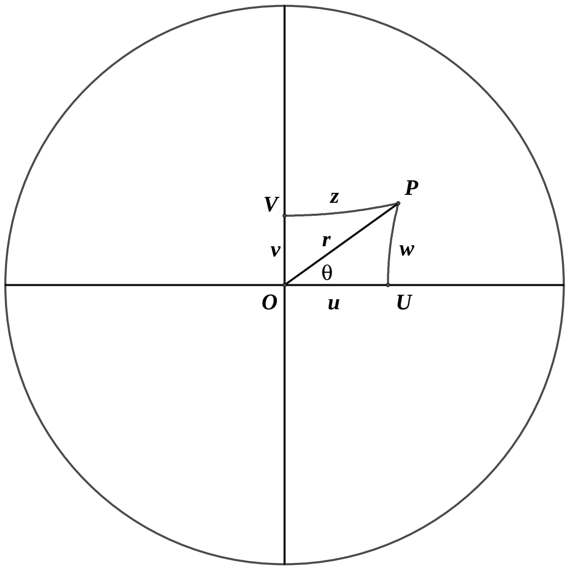


Figure 12.5: P in the Poincaré Disk

Every point has a unique ordered pair of Lobachevsky coordinates, and, conversely, every ordered pair of real numbers is the pair of Lobachevsky coordinates for some unique point. In Lobachevsky coordinates

1. for $a \neq 0$, $u = a$ is the equation of a line;
2. for $a \neq 0$, $w = a$ is the equation of a hypercycle;
3. $e^{-u} = \tanh w$ is an equation of the line in the first quadrant that is horoparallel to both axes.
4. $e^{u} = \cosh w$ is an equation of the horocycle with radius $\overrightarrow{OX}$.

Thus, a line does not have a linear equation in Lobachevsky coordinates, and a linear equation does not necessarily describe a line.

Every point has a unique ordered pair of axial coordinates. However, not every ordered pair of real numbers is a pair of axial coordinates. Let U and V be points on the axes with $V \neq 0$. Now the perpendiculars at U and V do not have to intersect. It is easy to see that they might be horoparallel or hyperparallel, especially by looking in the Poincaré model. If the two lines are limiting parallel (horoparallel) then that would make the segments OU and OV complementary segments. It can be shown then that these perpendiculars to the axes at U and V will intersect if and only if $|u| < |v|^*$. It then can be shown that (u, v) *are the axial coordinates of a point if and only if* $\tanh^2 u + \tanh^2 v < 1$.

Lemma 12.1 *With respect to a given frame*

i) Every point has a unique ordered pair of Beltrami coordinates, and (x, y) is an ordered pair of Beltrami coordinates if and only if $x^2 + y^2 < 1$.

ii) If the point P_1 has Beltrami coordinates (x_1, y_1) and point P_2 has Beltrami coordinates (x_2, y_2), then the distance $d_h(P_1P_2) = P_1P_2$ is given by the following *formulæ:*

$$\cosh P_1P_2 = \frac{1 - x_1x_2 - y_1y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}}$$

$$\tanh P_1P_2 = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (x_1y_2 - x_2y_1)^2}}{1 - x_1x_2 - y_1y_2}$$

iii) $Ax + By + C = 0$ is an equation of a line in Beltrami coordinates if and only if $A^2 + B^2 > C^2$, and every line has such an equation.

iv) Given an angle $\angle PQR$ and given that the Beltrami coordinates of P are (x_1, y_1), of Q are (x_2, y_2), and of R are (x_3, y_3), then the cosine of this angle is given by

$$\cos(\angle PQR) = \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) - (x_2y_1 - x_1y_2)(x_3y_1 - x_1y_3)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (x_1y_2 - x_2y_1)^2}\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 - (x_1y_3 - x_3y_1)^2}}.$$

v) If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are equations of two intersecting line in Beltrami coordinates and θ is the angle formed by their intersection, then

$$\cos\theta = \pm\frac{AD + BE - CF}{\sqrt{A^2 + B^2 - C^2}\sqrt{D^2 + E^2 - F^2}}.$$

In particular the lines are perpendicular if and only if $AD + BE = CF$.

vi) If (x_1, y_1) and (x_2, y_2) are the Beltrami coordinates of two distinct points, let $t_1 = \sqrt{1 - x_1^2 - y_1^2}$ and $t_2 = \sqrt{1 - x_2^2 - y_2^2}$. Then the midpoint of the segment joining the two points has Beltrami coordinates

$$\left(\frac{x_1t_2 + x_2t_1}{t_1 + t_2}, \frac{y_1t_2 + y_2t_1}{t_1 + t_2}\right)$$

and the perpendicular bisector of the two points has an equation

$$(x_1t_2 - x_2t_1)x + (y_1t_2 - y_2t_1)y + (t_1 - t_2) = 0.$$

vii) If $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are equations of lines in Beltrami coordinates and if $A_1B_2 = A_2B_1$, then the two lines are hyperparallel.

viii) Every cycle has an equation in Beltrami coordinates that is of the form

$$\sqrt{1 - x^2 - y^2} = ax + by + c.$$

1. *The cycle is a circle if and only if $-1 < a^2 + b^2 - c^2 < 0$ and $c > 0$.*
2. *The cycle is a horocycle if and only if $a^2 + b^2 - c^2 = 0$ and $c > 0$.*
3. *The cycle is a hypercycle if and only if $a^2 + b^2 - c^2 > 0$.*

In Poincaré coordinates (p, q)

$$C((p^2 + q^2) + 2Ap + 2Bq + C = 0$$

is an equation of a line if and only if $A^2 + B^2 > C^2$, and every line has such an equation.

The following is a representation of graph paper in the Poincaré disk model. Each line is 1/4 unit apart. The distances are measured along the u and v axes.

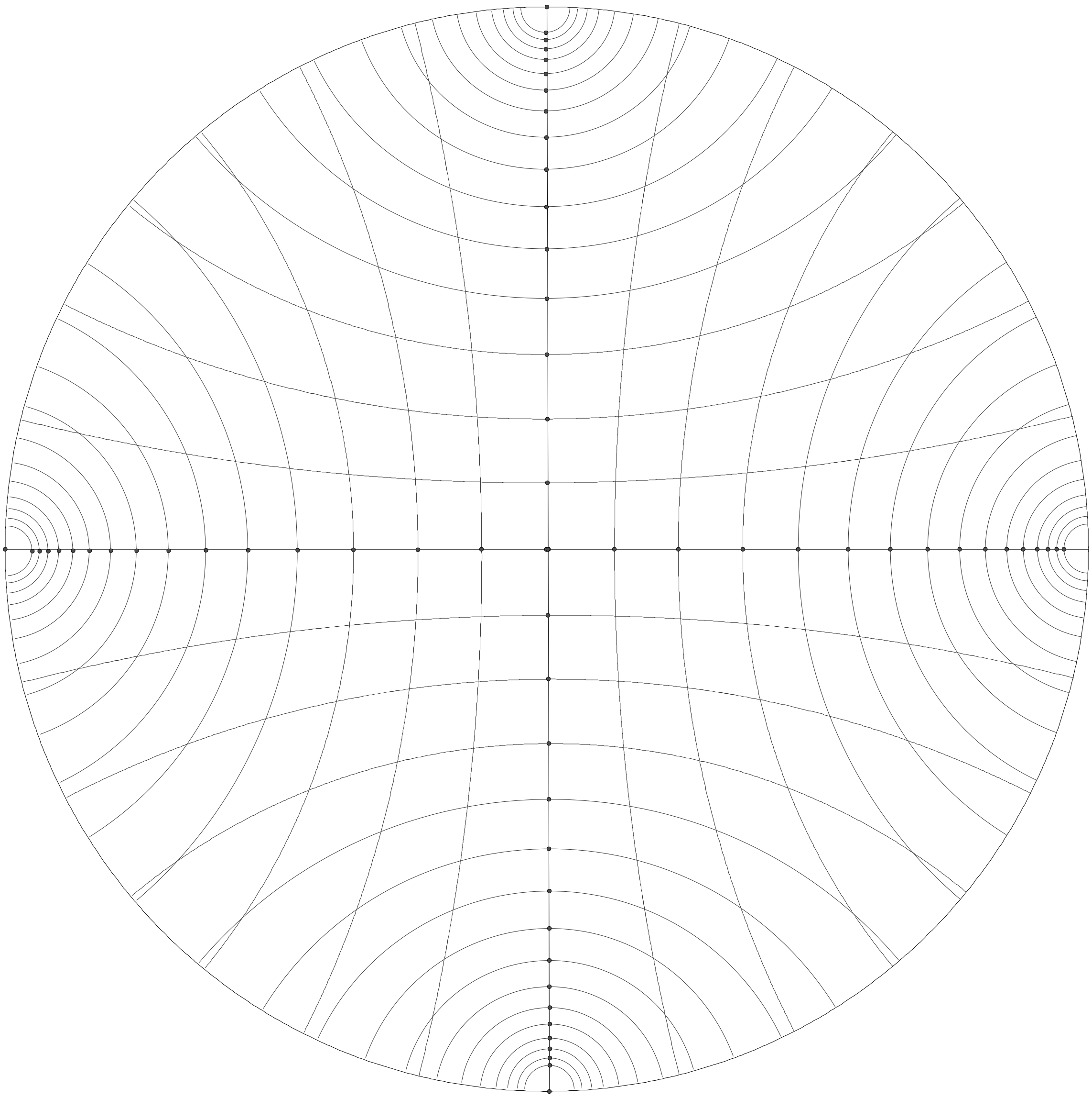


Figure 12.6: Hyperbolic Graph Paper