Chapter 2

The Rules of the Game

We have mentioned that Euclid gathered all that was known about geometry and gathered it together in his *Elements*. We note that from what we have found to date, the Greeks beginning with Thales of Miletus took a more theoretical view of geometry and worked to systematize geometric knowledge by showing that certain results follow logically from others.

2.1 Euclid's *Elements*

We believe that the work of the Greek geometers reached a peak with the appearance of Euclid's work. It appeared in 13 'books', where each book is more like one of a modern textbook's chapters.

Euclidean geometry is studied from three different perspectives that show themselves in high school geometry. The first and oldest of these is called the *traditional perspective*, or sometimes the *axiomatic* perspective. In this way of studying geometry we start, as did Euclid, with the axioms and postulates of Euclid. From this core of results, we can modify certain axioms or postulates and derive finite geometries, non-Euclidean geometries and other systems whose results can be compared to those that we deduced in Euclidean geometry.

A second perspective is credited to the German mathematician Felix Klein (1849–1925) who was looking for a manner in which to unify the study of geometry. He, together with the Swedish mathematician Sophus Lie (1842–1899), viewed geometry as studying properties of figures that are invariant under certain transformations. This was not really new, in some sense, since Euclid used the property of "superposition" in his books. This is called the *transformation perspective* of geometry. By modifying the properties selected to be invariant, we can generate different geometries — such as affine geometries, Euclidean geometry, non-Euclidean geometries, projective geometries, or even topology.

The third perspective also originates in the 19th century and is based on studying geometry as a vector space. This is called the *vector perspective*. It lends itself to a much more analytic study of geometry.

That being said, any of these perspectives can be approached either synthetically or analytically. The *synthetic* approaches tend to use numbers as rarely as possible and are based on a more axiomatic development of the geometry. The *analytic* approaches take advantage of the properties of numbers (and algebra) to deduce geometric properties.

2.2 Deduction and Proof

The process by which we develop *Geometry* is tried and true. We may not always start with exactly the same set of axioms or undefined terms, but we do know the path by which we will travel. We will generate theorems using the rules of "deductive logic." What does this mean?

In logic, especially the type that we will use in Geometry, we accept the following common notion.

LAW OF THE EXCLUDED MIDDLE A statement is either true or false, that is *P or not P*.

There is not other possibility. For example, if P is the proposition,

"*The grass is green.*"

then the Law of Excluded Middle holds that the logical statement

Either Socrates is mortal or Socrates is not mortal.

is true by virtue of its form alone.

Here is n example of an argument that depends on the Law of Excluded Middle.

Lemma 2.1 *There exist two irrational numbers a and b such that a^b is rational.*

PROOF: It is known that $\sqrt{2}$ is irrational. Consider the number

$$\sqrt{2}^{\sqrt{2}}.$$

Clearly this number is either rational or irrational. If it is rational, we are done. If it is irrational, then let

$$a = \sqrt{2}^{\sqrt{2}} \text{ and } b = \sqrt{2}.$$

Then

$$a^b = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

and 2 is certainly rational. This concludes the proof. ∎

In the above argument, the statement "*this number is either rational or irrational*" invokes the Law of Excluded Middle.

There are multivalued logics, or fuzzy logics, in which we do not take the stance that each statement must be true or false.

This stance that you can always determine the truth of a mathematical statement is true in some sense. If you agree with the assumptions in the statement or in the problem, and if valid reasoning has been used, the you must agree with the results. This property of truth based on assumptions and valid reasoning is a result of a basic underlying aspect of mathematical thinking — the *deductive process*, or *deduction*. In mathematics results that have been deduced from agreed upon statements, such as axioms, definitions or previously proven statements, using valid arguments of deduction can be thought of as ***true***. We call these results **theorems**. The deductive argument itself is called a **proof**.

Now, before we messed it all up by finding non-Euclidean geometries, theorems proved using deduction were usually considered to be **absolute truths**, or independent of the particular axioms that you chose for the geometry. We sometimes create different axioms sets because some results may be more accessible using this axiom set rather than another. We had thought that we always ended in the same place, so there was little concern. Now we know that the postulates that we chose *define* the mathematical system, so our results are **relative truths**, or true based on the assumptions of that system.

2.3 The Power of Deduction

It may seem hard to believe that this is such a powerful tool. However, it clearly trumps observation. In Euclidean geometry we prove that the sum of the angles of a triangle is $180°$. This isn't just the triangles you can draw on your paper, or the triangles that appear on the screen in a Dynamic Geometry System (DGS), or triangles that have whole number angle measures. It is **EVERY** triangle. You cannot find every triangle and measure it — and even if you could, you still could not measure it precisely enough. We know that every triangle in Euclidean geometry has the same angle sum — every one!

There are times when we use deduction to "prove the obvious" such as that *the base angles of an isosceles triangle are congruent*. However, we must remember that what may seem obvious in one case may not be so obvious in the next. Also, sometimes what seems "obvious" may turn out not to be true. Another strength is that by deduction we can prove theorems to be true even when they are hard to believe.

Another aspect of deduction is that it not only can show a statement to be true, but also can indicate why it is true. A fourth aspect of the power of deduction is that it provides a universally accepted criterion for the establishment of mathematical truth. Although some philosophers may debate about the foundations of mathematics and some mathematicians question whether a proof in which computers did much of the work (such as the solution to the Four Color Theorem) constitutes a proof, or whether a complicated proof has a gap, there is universal agreement on the principles behind deductive proof. Consequently, when a new theorem is proved (consider the proof of Fermat's Last Theorem or, more recently, the solution of the Poincaré conjecture), no one goes to the lab to check the results. Only the argument is checked to see that the steps were valid based on the suppositions.

2.4 All Power Corrupts

With power comes responsibility. It must be used carefully. This is not always the case. Geometry is especially suspect because we so much want to rely on drawings to "see" what is going on. These can often lead one astray!!

Theorem 2.1 *Every triangle is isosceles.*

Proof: Given $\triangle ABC$ with $AC \neq BC$.

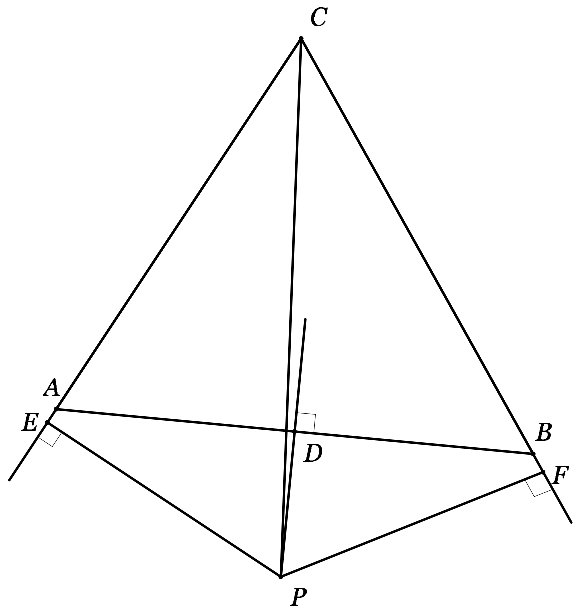
1. Construct the bisector of $\langle C$. Call it ℓ. ℓ is not perpendicular to $\overline{AB}$ because $AC \neq BC$.

2. Construct the perpendicular bisector of segment AB. Call it m and let $D = AB \cap m$. Since ℓ and m are not both perpendicular to AB, ℓ and m must intersect in a point, P.

3. Construct the perpendicular from P to $\overleftrightarrow{BC}$, and let F be the foot of P in $\overleftrightarrow{BC}$. Likewise, construct the perpendicular from P to $\overleftrightarrow{AC}$ and let E be the foot of P in $\overleftrightarrow{AC}$.

4. $\langle ACP \cong \langle BCP$, since $\overleftrightarrow{CP} = \ell$ is the bisector of $\langle C$.

5. $\langle CEP \cong \langle CFP$, since they are both right angles.



6. $\triangle CEP \cong \triangle CFP$ by AAS.
7. $EP \cong FP$ and $CE \cong CF$ because 'CPCTC'[1]
8. So, $AP \cong BP$ since a point on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.
9. $\angle AEP \cong \angle BFP$ since they are both right angles.
10. *CLAIM*: $EA \cong FB$.
    (a) Suppose that $EA > BF$, then there is a point A' on EA, different from A, with $EA' \cong BF$. Then $\angle A'EP$ is a right angle.
    (b) $\angle AA'P$ is obtuse by the Exterior Angle Theorem.
    (c) $\triangle FPB \cong \triangle EPA'$ by SAS.
    (d) $PA' \cong PB$ — CPCTC
    (e) $PA' \cong PA$

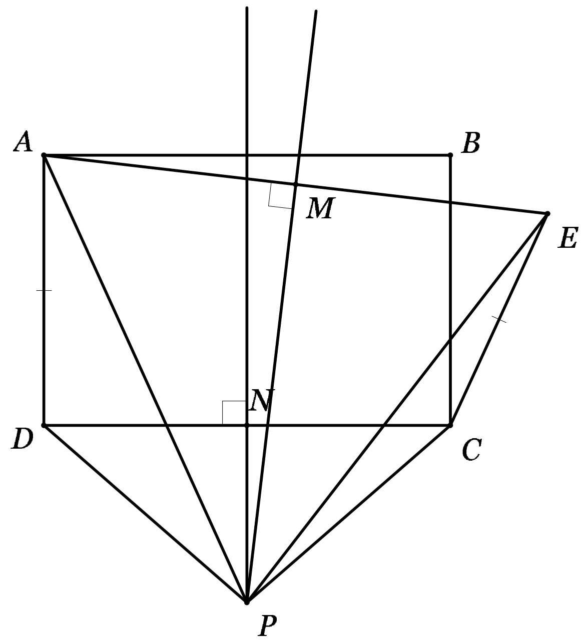
    This is impossible, so $EA \cong FB$.
11. $AC \cong BC$ since $CE \cong CF$ and $EA \cup AC = CE$ and $FB \cup BC = CF$.

Therefore $\triangle ABC$ is isosceles. ∎

Okay, what is wrong? If nothing is wrong, then the result has to be true, yet you can probably easily think of a triangle that is not isosceles. Did we use an incorrect result? No. We any steps logically invalid? No. Yet something is wrong.

Here is another conundrum — you need to find the mistake.

Theorem 2.2 *A right angle has the same measure as an obtuse angle.*



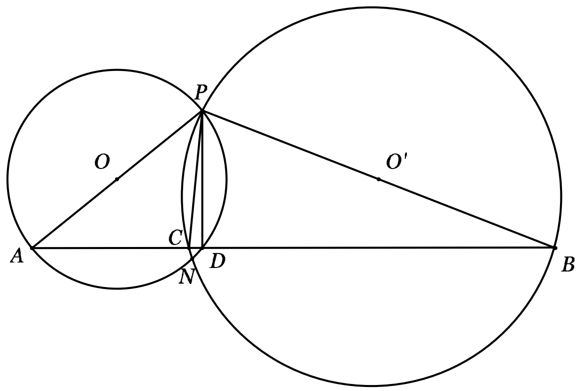
PROOF: Construct a rectangle $\square ABCD$. Choose a point E not on the rectangle so that $AD \cong CD$. Construct the perpendicular bisector of AE and call it ℓ. Construct the perpendicular bisector of CD and call it m. Then $\ell \cap m \neq \emptyset$. Call the point of intersection P. Let M be the midpoint of AE and let N be the midpoint of CD. Construct segments DP, EP, AP, and CP.

1. $AP \cong EP$, since P is on the perpendicular bisector of AE.
2. $DP = CP$ since P is on the perpendicular bisector of CD.
3. By construction $AD \cong CE$.
4. Thus, $\triangle ECP \cong \triangle ADP$ by the SSS Congruence criterion.
5. Therefore $\angle ECP \cong \angle ADP$ by CPCTC.
6. Since $DP \cong CP$, $\triangle DNP \cong \triangle CNP$, by the SSS Congruence criterion.
7. $\angle DCP \cong \angle CDP$ by CPCTC.
8. Now, $\angle ECP = \angle DCP + \angle ECD$ and $\angle ADP = \angle CDP + \angle ADC$.
9. Thus, $\angle ECD \cong \angle ADC$.
10. Since E lies outside the rectangle, $\angle ECD > \angle BCD$ so it is obtuse.
11. $\angle ACD$ is an angle of the rectangle, so it is a right angle.
12. We have $\angle ECD \cong \angle ADC$ so a right angle is congruent to an obtuse angle.

[1] corresponding parts of congruent triangles are congruent

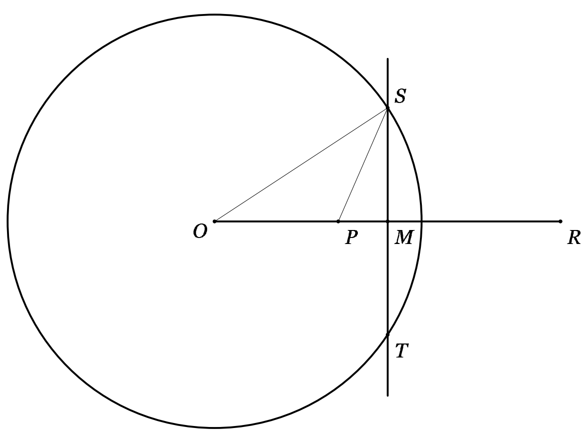
This completes the proof. ∎

Theorem 2.3 *Two distinct perpendiculars can be drawn to a given line from a given external point.*



PROOF: Draw any two circles centered at O and O', intersecting at points P and N. Draw diameters PA and PB and then draw AB intersecting the circles at C and D. $\angle PDA$ and $\angle PCB$ are right angles because they are inscribed in semicircles. Thus PC and PD are both perpendicular to AB. ∎

Theorem 2.4 *Every point inside a circle is on the circle.*



PROOF: Consider the circle γ with center O and let P be inside the circle. Choose a point R on ray $\overrightarrow{OP}$ so that $(OP)(OR) = r^2$, where r is the radius of γ. Let the perpendicular bisector of PR intersect the circle at points S and T and let M be the midpoint of PR.

$$\begin{aligned} OP &= OM - MP \\ OR &= OM + MR = OM + MP \\ (OP)(OR) &= (OM - MP)(OM + MP) \\ &= (OM)^2 - (MP)^2 \end{aligned}$$

By the Pythagorean Theorem

$$\begin{aligned}
(OM)^2 + (MS)^2 &= (OS)^2 \\
(OM)^2 &= (OS)^2 - (MS)^2 \\
(MP)^2 + (MS)^2 &= (PS)^2 \\
(MP)^2 &= (PS)^2 - (MS)^2 \\
(OP)(OR) &= (OM)^2 - (MP)^2 = [(OS)^2 - (MS)^2] - [(PS)^2 - (MS)^2] \\
&= (OS)^2 - (PS)^2
\end{aligned}$$

But $(OS)^2 = r^2 = (OP)(OR)$

$$(OP)(OR) = (OP)(OR) - (PS)^2$$

Therefore, $PS = 0$ and P must be on the circle! ∎

The purpose of this section is to show us that while the power of deduction is a very powerful tool, we can misuse it by relying too much on diagrams. This means that we must question many of the diagrams in proofs. We should have an axiomatic or logical reason for a diagram to be represented in a proof. It can be a guideline, but it cannot serve as a proof itself.