Chapter 3

Euclidean Geometry

3.1 Euclid's Axioms for Geometry

I mentioned Euclid's Axioms earlier. Now, we want to be more careful in the way that we frame the axioms and make our definitions. This is the basis with which we must work for the rest of the semester. If we do a bad job here, we are stuck with it for a long time.

Since we do not want to have to second guess everything that we prove, we will want to agree on some facts that are absolute and unquestionable. These should be accepted by all and should be easily stated. These will be our axioms. Euclid chose to work with five axioms. (Hilbert in his later work chose to work with 16 axioms.)

Postulate 1: We can draw a unique line segment between any two points.

Postulate 2: Any line segment can be continued indefinitely.

Postulate 3: A circle of any radius and any center can be drawn.

Postulate 4: Any two right angles are congruent.

Postulate 5: Given a line ℓ and a point P not on ℓ, there exists a unique line ℓ_2 through P which does not intersect ℓ.[1]

What assumptions have we made here? First of all, we have assumed that a set of points, called the *Euclidean plane* exists. With this assumption comes the concept of length, of lines, of circles, of angular measure, and of congruence. It also assumes that the plane is two-dimensional. All this in five little sentences.

Let's consider what Hilbert does in his choices, and then what Birkhoff chose.

3.1.1 Hilbert's Axioms for Neutral Geometry

GROUP I : Incidence Axioms

I–1: For every point P and for every point Q not equal to P there exists a unique line ℓ that passes through P and Q.

I–2: For every line ℓ there exist at least two distinct points incident with ℓ.

I–3: There exist three distinct points with the property that no line is incident with all three of them.

[1]This is actually Playfair's postulate. We will give the statement of Euclid later.

GROUP II : Betweeness Axioms

B–1: If $A * B * C$[2], then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

B–2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A * B * D$, $B * C * D$, and $B * D * E$.

B–3: If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

B–4: (PLANE SEPARATION AXIOM) For every line ℓ and for any three points A, B, and C not lying on ℓ:

(a) if A and B are on the same side of ℓ and B and C are on the same side of ℓ, then A and C are on the same side of ℓ.

(b) if A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

GROUP III : Congruence Axioms

C–1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

C–2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

C–3: If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

C–4: Given any $\angle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\angle B'A'C' \cong \angle BAC$.

C–5: If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.

C–6: (*SAS*) If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

GROUP IV: Continuity Axioms

ARCHIMEDES' AXIOM: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

DEDEKIND'S AXIOM: Suppose that the set of all points on a line ℓ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and *vice versa*. Then there is a unique point, O, lying on ℓ such that $P_1 * O * P_2$ if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1, P_2$.

[2] $A * B * C$ denotes that B lies between A and C

(The following two Principles follow from Dedekind's Axiom, yet are at times more useful.)

Circular Continuity Principle: If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Elementary Continuity Principle: If one endpoint of a segment is inside a circle and the other outside the circle, then the segment intersects the circle.

Hilbert also used five undefined terms: **point, line, incidence, betweenness, and congruence**.

3.1.2 Birkhoff's Axioms for Neutral Geometry

The setting for these axioms is the "Absolute (or Neutral) Plane". It is universal in the sense that all points belong to this plane. It is denoted by A^2.

Axiom 1: There exist nonempty subsets of A^2 called "*lines*," with the property that each two points belong to *exactly* one line.

Axiom 2: Corresponding to any two points A, $B \in A^2$ there exists a *unique* number $d(AB) = d(BA) \in \mathbb{R}$, the *distance* from A to B, which is 0 if and only if $A = B$.

Axiom 3: (*Birkhoff Ruler Axiom*) If k is a line and $\mathbb{R}$ denotes the set of real numbers, there exists a *one-to-one correspondence* $(X \leftrightarrow x)$ between the points $X \in k$ and the numbers $x \in \mathbb{R}$ such that

$$d(A, B) = |a - b|$$

where $A \leftrightarrow a$ and $B \leftrightarrow b$.

Axiom 4: For each line k there are exactly two nonempty *convex* sets R' and R'' satisfying

(a) $A^2 = R' \cup k \cup R''$

(b) $R' \cap R'' = \phi$, $R' \cap k = \phi$, and $R'' \cap k = \phi$. That is, they are *pairwise disjoint*.

(c) If $X \in R'$ and $Y \in R''$ then $XY \cap k \neq \phi$.

Axiom 5: To each angle $\angle ABC$ there exists a unique real number x with $0 \leq x \leq 180$ which is the (*degree*) measure of the angle

$$x = \angle ABC^\circ.$$

Axiom 6: If $\overrightarrow{BD} \subset \text{Int}\,(\angle ABC)$, then

$$\angle ABD^\circ + \angle DBC^\circ = \angle ABC^\circ.$$

Axiom 7: If $\overrightarrow{AB}$ is a ray in the edge, k, of an open half plane $H(k; P)$ then there exist a *one-to-one correspondence* between the open rays in $H(k; P)$ emanating from A and the set of real numbers between 0 and 180 so that if $\overrightarrow{AX} \leftrightarrow x$ then

$$\angle BAX^\circ = x.$$

Axiom 8: (*SAS*) If a correspondence of two triangles, or a triangle with itself, is such that two sides and the angle between them are respectively congruent to the corresponding two sides and the angle between them, the correspondence is a congruence of triangles.

3.2 Review of Results from Euclidean Geometry

We will prove some of these results later, but to start let's review some of the results from Euclidean Geometry that you (and I) may (or may not) recall.

Instead of listing them as we would prove them, we will group them by topics.

3.2.1 Quadrilaterals

1. Methods of proving that a quadrilateral is a *parallelogram.*
   To prove that a quadrilateral is a parallelogram, prove that:
   (a) both pairs of opposite sides are parallel;
   (b) both pairs of opposite sides are congruent;
   (c) one pair of sides are both congruent and parallel;
   (d) both pairs of opposite angles are congruent;
   (e) one pair of opposite angles are congruent and one pair of opposite sides are parallel;
   (f) the diagonals bisect each other.
2. Methods of proving that a quadrilateral is a *rectangle.*
   To prove that a quadrilateral is a rectangle, prove that:
   (a) it has four right angles;
   (b) it is a parallelogram with one right angle;
   (c) it is a parallelogram with congruent diagonals.
3. Methods of proving that a quadrilateral is a *rhombus.*
   To prove that a quadrilateral is a rhombus, prove that:
   (a) it has four congruent sides;
   (b) it is a parallelogram with two consecutive sides congruent;
   (c) it is a parallelogram in which a diagonal bisects an angle of the parallelogram;
   (d) it is a parallelogram with perpendicular diagonals.
4. Methods of proving that a quadrilateral is a *square.*
   To prove that a quadrilateral is a a square, prove that:
   (a) it is a rectangle with two consecutive sides congruent;
   (b) it is a rectangle with a diagonal bisecting one of its angles;
   (c) it is a rectangle with perpendicular diagonals;
   (d) it is a rhombus with one right angle;
   (e) it is a rhombus with congruent diagonals.
5. Methods for proving that a trapezoid[3] is isosceles.
   To prove that a trapezoid is isosceles, prove that:

[3]Our definition of trapezoid is a quadrilateral with exactly one pair of opposite sides parallel. Some authors define a trapezoid to be a quadrilateral with at least one pair of opposite sides parallel.

(a) its nonparallel sides are congruent;

(b) the base angles are congruent;

(c) the opposite angles are supplementary;

(d) its diagonals are congruent.

3.2.2 Midline of a Triangle

1. The *midline* of a triangle is the line segment joining the midpoints of two sides of the triangle.

2. The midline of a triangle is parallel to the third side of the triangle.

3. The midline of a triangle is half as long as the third side of the triangle.

4. If a line containing the midpoint of one side of a triangle is parallel to a second side of the triangle, then it also contains the midpoint of the third side of the triangle.

3.2.3 Similarity

1. When a line is parallel to one side of a triangle

    (a) if a line parallel to one side of a triangle intersects the other two sides, then it divides these two sides proportionally;

    (b) if a line divides two sides of a triangle proportionally, then the line is parallel to the remaining side of the triangle.

2. Proportionality involving *angle bisectors*

    (a) An interior angle bisector of any triangle divides the side of the triangle opposite the angle into segments proportional to the adjacent sides. In Figure 3.1, AD is the angle bisector and

$$\frac{CD}{DB} = \frac{CA}{AB}.$$

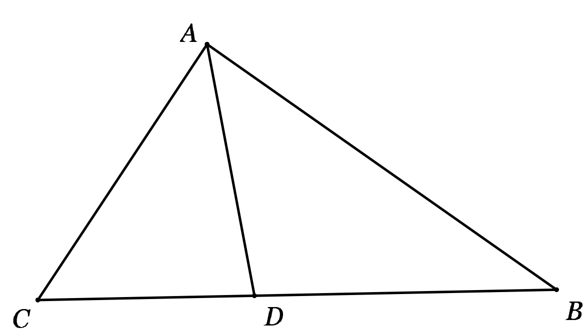

Figure 3.1:

(b) An exterior angle bisector of a triangle determines, with each of the other vertices, segments along the line containing the opposite side of the triangle that are proportional to the two remaining sides. In Figure 3.2, AD is an exterior angle bisector and

$$\frac{CD}{DB} = \frac{CA}{AC}.$$

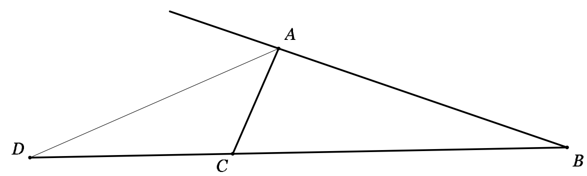

Figure 3.2:

3. Methods of proving triangles similar.
   (a) If two triangles are similar to the same triangle, or to similar triangles, then the triangles are similar to one another.
   (b) IF two pairs of corresponding angles of two triangles are congruent, then the triangles are similar.
   (c) If two sides of one triangle are proportional to two sides of another triangle and the angles included by those sides are congruent, then the triangles are similar.
   (d) If the corresponding sides of two triangles are proportional, the the two triangles are similar.
4. *Mean proportionals* in a right triangle.
   (a) The altitude to the hypotenuse of a right triangle divides the hypotenuse so that either leg is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to that leg.
   (b) The altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

3.2.4 Pythagorean Theorem

1. The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse.
2. *Converse of the Pythagorean Theorem*: If the sum of the squares of the lengths of two sides of a triangle equals the square of the length of the third side, then the angle opposite this third side is a right angle.
3. In an isosceles right triangle:
   (a) the hypotenuse is $\sqrt{2}$ times as long as a leg.
   (b) either leg is $1/\sqrt{2}$ times as long as the hypotenuse.
4. In a 30–60–90 triangle:

(a) the side opposite the angle of measure 30° is half as long as the hypotenuse;

(b) the side opposite the angle of measure 60° is $\sqrt{3}/2$ times as long as the hypotenuse;

(c) the hypotenuse is $2/\sqrt{3}$ times as long as the side opposite the angle of measure 60°;

(d) the longer leg is $\sqrt{3}$ times as long as the shorter leg.

5. Pythagorean identities

   (a) In an acute triangle, the square of the length of any side is less than the sum of the squares of the lengths of the two remaining sides.

   $$a^2 + b^2 > c^2$$

   (b) In an obtuse triangle, the square of the length of the longest side is greater than the sum of the squares of the lengths of the two shorter sides.

   $$a^2 + b^2 < c^2$$

6. *Extension of the Pythagorean theorem*: If similar polygons are constructed on the sides of the right triangle (with corresponding sides used for a side of the right triangle), then the area of the polygon on the hypotenuse equals the sum of the areas of the polygons on the legs.

3.2.5 Circle Relationships

1. Angle measurement with a circle.

   (a) The measure of an inscribed angle is one-half the measure of its intercepted arc.

   (b) The measure of an angle formed by a tangent and a chord of a circle is one-half the measure of its intercepted arc.

   (c) The measure of an angle formed by two chords intersecting in a point in the interior of a circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

   (d) The measure of an angle formed by two secants of a circle intersecting in a point in the exterior of the circle is equal to one-half the difference of the measures of the intercepted arcs.

   (e) The measure of an angle formed by a secant and a tangent to a circle intersecting in a point in the exterior of the circle is equal to one-half the difference of the measures of the intercepted arcs.

   (f) The measure of an angle formed by two tangents to a circle is equal to one-half the difference of the measures of the intercepted arcs.

   (g) The sum of the measure of an angle formed by two tangents to a circle and the measure of the closer intercepted arc is 180°.

2. Methods of proving four point *concyclic*[4]

[4] A cyclic quadrilateral is a quadrilateral whose vertices are concyclic, that is lie on the same circle.

(a) If one side of a quadrilateral subtends congruent angles at the two consecutive vertices, then the quadrilateral is cyclic.

(b) If a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

3. Tangent, secant, and chord segments

   (a) Two tangent segments that have the same endpoint in the exterior of the circle to which they are tangent are congruent.

   (b) If a secant segment and a tangent segment to the same circle share an endpoint in the exterior of the circle, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external segment

   $$(AP)^2 = (PC)(PB).$$

   (c) IF two secant segments of the same circle share an endpoint in the exterior of the circle, then the product of the lengths of one secant segment and its external segment equals the product of the lengths of the other secant segment and its external segment

   $$(AP)(BP) = (DP)(CP).$$

   (d) If two chords intersect in the interior of a circle, thus determining two segments in each chord, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord

   $$(AP)(BP) = (DP)(CP)$$

3.2.6 Concurrency

1. The perpendicular bisectors of the sides of a triangle are concurrent at a point that is the *center of the circumscribed circle*, the *circumcenter* of the triangle.

2. The lines containing the three altitudes of the triangle are concurrent at a point called the *orthocenter* of the triangle.

3. The medians of a triangle are concurrent at a point of each median located two-thirds of the way from the vertex to the opposite side. This point is called the *centroid* of the triangle and is the center of gravity of the triangle.

4. The angle bisectors of a triangle are concurrent at a point that is the *center of the inscribed circle*, or the *incenter* of the triangle.

3.2.7 Inequalities

1. The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

2. IF two sides of a triangle are not congruent, then the angles opposite those sides are not congruent, the angle with the greater measure being opposite the longer side.

3. IF two angles of a triangle are not congruent, then the sides opposite those angles are not congruent, the longer side being opposite the angle with greater measure.

4. (*The Triangle Inequality*) The sum of the lengths of any tow sides of a triangle is greater than the length of the third side.

5. If two sides of a triangle are congruent respectively to two sides of a second triangle and the measure of the included angle of the first triangle is greater than the measure of the the included angle of the second triangle, then the measure of the third side of the first triangle is greater than the measure of the third side of the second triangle.

6. If two sides of a triangle are congruent respectively to two sides of a second triangle and the measure of the third side of the first triangle is greater than the measure of the the third side of the second triangle, then the measure of the included angle of the first triangle is greater than the measure of the included angle of the second triangle.

7. In an acute triangle, the square of the length of any side is less than the sum of the squares of the lengths of the two remaining sides.

8. In an obtuse triangle, the square of the length of the longest side is greater than the sum of the squares of the lengths of the two shorter sides.

3.2.8 Area

1. The area of a square equals the square of the length of a side

$$A = s^2.$$

2. The are of a square equals one-half the square of the length of one of its diagonals

$$A = \frac{1}{2}d^2.$$

3. The area of a right triangle equals one-half the product of the lengths of its legs

$$K = \frac{1}{2}\ell_1 \cdot \ell_2.$$

4. If two triangles have congruent bases, then the ratio of their areas equals the ratio of the lengths of their altitudes.

5. If two triangles have congruent altitudes, then the ratio of their areas equals the ratio of the lengths of their bases.

6. The are of any triangle equals one-half the product of the lengths of any two sides multiplied by the sine of the include angle

$$K = \frac{1}{2}ab\sin\angle C.$$

7. The ratio of the areas of two triangles that have one pair of congruent corresponding angles equals the ratio of the products of the lengths of the pairs of sides that include the angles
$$\text{If } \angle B \cong \angle E, \text{ then } \frac{\text{area}\triangle ABC}{\text{area}\triangle DEF} = \frac{(AB)(BC)}{(DE)(EF)}.$$

8. The area of an equilateral triangle equals $\sqrt{3}/4$ times the square of the length of a side
$$K = \frac{s^2\sqrt{3}}{4}.$$

9. The area of an equilateral triangle equals $\sqrt{3}/3$ times the square of the length of an altitude
$$K = \frac{h^2\sqrt{3}}{3}.$$

10. (*Heron's Formula*) The are of any triangle with sides of length a, b, and c is
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
where $s = \dfrac{a+b+c}{2}$ is the semiperimeter.

11. The area of a parallelogram equals the product of the lengths of a base and the altitude to that base
$$A = b \cdot h.$$

12. The area of a rhombus equals one-half the product of the lengths of its diagonals
$$A = \frac{1}{2}(d_1 \cdot d_2).$$

13. The area of a trapezoid equals one-half the product of the length of the altitude and the sum of the lengths of the bases
$$A = h\frac{a+b}{2}.$$

14. The area of a regular polygon equals one-half the product of the lengths of the apothem and the perimeter
$$A = \frac{1}{2}a \cdot p.$$

15. The area of a sector with radius r and a central angle of measure n degrees equals
$$\frac{n}{360} \cdot \pi r^2.$$

16. The area of a sector with radius r and a central angle of measure θ radians equals
$$\frac{\theta}{2} \cdot r^2.$$

17. The ratio of the areas of two similar triangles equals the square of the ratio of similitude.

18. The ratio of similitude of any pair of similar triangles equals the square root of the ratio of their areas.

19. The ratio of the areas of two similar polygons equals the square of their ratio of similitude.

20. The ratio of similitude of any pair of similar polygons equals the square root of the ratio of their areas.